SIMPLICITY OF A_n

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1. Introduction

A finite group is called *simple* when it is nontrivial and its only normal subgroups are the trivial subgroup and the whole group.

For instance, a finite group of prime order is simple, since it in fact has no nontrivial proper subgroups at all (normal or not). A finite abelian group G not of prime order is not simple: let p be a prime factor of $|G|$, so G contains a subgroup of order p, which is normal since G is abelian and is proper since $|G| > p$. Thus, the abelian finite simple groups are the groups of prime order.

When $n \geq 3$ the group S_n is not simple since it has the normal subgroup A_n of index 2.

Theorem 1.1. *For $n \geq 5$, the group A_n is simple.*

This is due to Camille Jordan [6, p. 66] in 1870. The special case $n = 5$ goes back to Galois. The restriction $n \geq 5$ is optimal, since A_4 is not simple: it has the normal subgroup $\{(1), (12)(34), (13)(24), (14)(23)\}$. The group A_3 is simple, since it has order 3, and the groups A_1 and A_2 are trivial.

We will give *five* proofs of Theorem 1.1. Section 2 includes some preparatory material and later sections give the proofs of Theorem 1.1. In the final section, we give a quick application of the simplicity of alternating groups and some references for further proofs not treated here.

2. Preliminary lemmas

We need three lemmas: two are about alternating groups and one is about symmetric groups on n letters for $n \geq 5$.

Lemma 2.1. *For $n \geq 3$, A_n is generated by 3-cycles. For $n \geq 5$, A_n is generated by permutations of type $(2, 2)$.*

Proof. The identity (1) is (123)(132), which is a product of 3-cycles. Now pick a non-identity element of A_n, say σ and write it as a product of transpositions in S_n:

$$\sigma = \tau_1\tau_2 \cdots \tau_r.$$

The left side has sign 1 and the right side has sign $(-1)^r$, so r is even. Therefore we can collect the products on the right into successive transpositions $\tau_i\tau_{i+1}$, where $i = 1, 3, \ldots$ is odd. We will now show every product of two transpositions in S_n is a product of two 3-cycles, so σ is a product of 3-cycles.

<u>Case 1</u>: τ_i and τ_{i+1} are equal. Then $\tau_i\tau_{i+1} = (1) = (123)(132)$, so we can replace $\tau_i\tau_{i+1}$ with a product of two 3-cycles.

<u>Case 2</u>: τ_i and τ_{i+1} have exactly one element in common. Let the common element be a, so we can write $\tau_i = (ab)$ and $\tau_{i+1} = (ac)$, where $b \neq c$. Then

$$\tau_i\tau_{i+1} = (ab)(ac) = (acb) = (abc)(abc),$$

so we can replace $\tau_i\tau_{i+1}$ with a product of two 3-cycles.

Case 3: τ_i and τ_{i+1} have no elements in common. This means τ_i and τ_{i+1} are disjoint, so we can write $\tau_i = (ab)$ and $\tau_{i+1} = (cd)$ where a, b, c, d are distinct (so $n \geq 4$). Then

$$\tau_i\tau_{i+1} = (ab)(cd) = (ab)(bc)(bc)(cd) = (bca)(cdb) = (abc)(bcd),$$

so we can replace $\tau_i\tau_{i+1}$ with a product of two 3-cycles.

To show for $n \geq 5$ that An_n is generated by permutations of type $(2, 2)$, it suffices to write each 3-cycle (abc) in terms of such permutations. Pick $d, e \notin \{a, b, c\}$ (we can do this since $n \geq 5$). Then note

$$(abc) = (ab)(de)(de)(bc)$$

and the permutations $(ab)(de)$ and $(de)(bc)$ have type $(2, 2)$ since a, b, c, d, e are distinct. □

The 3-cycles in S_n are all conjugate in S_n, since permutations of the same cycle type in S_n are conjugate. Are 3-cycles conjugate in A_n? Not when $n = 4$: (123) and (132) are not conjugate in A_4. But for $n \geq 5$ we do have conjugacy in A_n.

Lemma 2.2. *For $n \geq 5$, all 3-cycles in A_n are conjugate in A_n.*

Proof. We show every 3-cycle in A_n is conjugate within A_n to (123). Let σ be a 3-cycle in A_n. It can be conjugated to (123) in S_n:

$$(123) = \pi\sigma\pi^{-1}$$

for some $\pi \in S_n$. If $\pi \in A_n$ we're done. Otherwise, let $\pi' = (45)\pi$, so $\pi' \in A_n$ and

$$\pi'\sigma\pi'^{-1} = (45)\pi\sigma\pi^{-1}(45) = (45)(123)(45) = (123).$$ □

Example 2.3. The 3-cycles (123) and (132) are not conjugate in A_4. But in A_5 we have

$$(132) = \pi(123)\pi^{-1}$$

for $\pi = (45)(12) \in A_5$.

Most proofs of the simplicity of the groups A_n are based on Lemmas 2.1 and 2.2. The basic argument is this: show each nontrivial normal subgroup $N \lhd A_n$ contains a 3-cycle, so N contains every 3-cycle by Lemma 2.2, and therefore N is A_n by Lemma 2.1.

The next lemma will be used in our fifth proof of the simplicity of alternating groups.

Lemma 2.4. *For $n \geq 5$, the only nontrivial proper normal subgroup of S_n is A_n. In particular, the only subgroup of S_n with index 2 is A_n.*

Proof. The last statement follows from the first since every subgroup of index 2 is normal.

Let $N \lhd S_n$ with $N \neq \{(1)\}$. We will show $A_n \subset N$, so $N = A_n$ or S_n.

Pick $\sigma \in N$ with $\sigma \neq (1)$. That means there is an i with $\sigma(i) \neq i$. Pick $j \in \{1, 2, \ldots, n\}$ so $j \neq i$ and $j \neq \sigma(i)$. Let $\tau = (ij)$. Then

$$\sigma\tau\sigma^{-1}\tau^{-1} = (\sigma(i)\ \sigma(j))(ij).$$

Since $\sigma(i) \neq i$ or j and $\sigma(i) \neq \sigma(j)$ (why?), the 2-cycles $(\sigma(i)\ \sigma(j))$ and (ij) are unequal, so their product is not the identity. That shows $\sigma\tau \neq \tau\sigma$.

Since $N \lhd S_n$, $\sigma\tau\sigma^{-1}\tau^{-1}$ lies in N. By construction, $\sigma(i) \neq i$ or j. If $\sigma(j) \neq i$ or j, then $(\sigma(i)\ \sigma(j))(ij)$ has type $(2, 2)$. If $\sigma(j) = i$ or j, $(\sigma(i)\ \sigma(j))(ij)$ is a 3-cycle. Thus N contains a permutation of type $(2, 2)$ or a 3-cycle. Since $N \lhd S_n$, N contains all permutations of type $(2, 2)$ or all 3-cycles. In either case, this shows (by Lemma 2.1) that $N \supset A_n$. □

Remark 2.5. There is an analogue of Lemma 2.4 for the "countable" symmetric group S_∞ consisting of all permutations of $\{1, 2, 3, \dots\}$. A theorem of Schreier and Ulam (1933) says the only nontrivial proper normal subgroups of S_∞ are $\cup_{n\geq 1} S_n$ and $\cup_{n\geq 1} A_n$, which are the subgroup of permutations fixing all but a finite number of terms and its subgroup of even permutations.

Remark 2.6. By Lemma 2.4, for $n \geq 5$ each homomorphic image of S_n not isomorphic to S_n has order 1 or 2. So there is no surjective homomorphism $S_n \to \mathbf{Z}/(m)$ for $m \geq 3$.

Theorem 2.7. *For $n \geq 5$, no subgroup of S_n has index strictly between 2 and n. Moreover, each subgroup of index n in S_n is isomorphic to S_{n-1}.*

Proof. Let H be a proper subgroup of S_n and let $m := [S_n : H]$, so $m \geq 2$. If $m = 2$ then $H = A_n$ by Lemma 2.4. If $m < n$ then we will show $m = 2$. The left multiplication action of S_n on S_n/H gives a group homomorphism

$$\varphi\colon S_n \to \mathrm{Sym}(S_n/H) \cong S_m.$$

By hypothesis $m < n$, so φ is not injective. Let K be the kernel of φ, so $K \subset H$ and K is nontrivial. Since $K \lhd S_n$, Lemma 2.4 says $K = A_n$ or S_n. Since $K \subset H$, we get $H = A_n$ or S_n, which implies $m = 2$. Therefore we can't have $2 < m < n$.

Now let H be a subgroup of S_n with index n. Consider the left multiplication action of S_n on S_n/H. This is a homomorphism $\ell\colon S_n \to \mathrm{Sym}(S_n/H)$. Since S_n/H has order n, $\mathrm{Sym}(S_n/H)$ is isomorphic to S_n. The kernel of ℓ is a normal subgroup of S_n that lies in H (why?). Therefore the kernel has index at least n in S_n. Since the only normal subgroups of S_n are 1, A_n, and S_n, the kernel of ℓ is trivial, so ℓ is an isomorphism. What is the image $\ell(H)$ in $\mathrm{Sym}(S_n/H)$? Since $gH = H$ if and only if $g \in H$, $\ell(H)$ is the group of permutations of S_n/H that fixes the "point" H in S_n/H. The subgroup fixing a point in a symmetric group isomorphic to S_n is isomorphic to S_{n-1}. Therefore $H \cong \ell(H) \cong S_{n-1}$. □

Theorem 2.7 is false for $n = 4$: S_4 contains the dihedral group of order 8 as a subgroup of index 3. An analogue of Theorem 2.7 for alternating groups will be given in Section 8; its proof will use the simplicity of the alternating groups.

Remark 2.8. That S_n has no subgroup with index strictly between 2 and n when $n \geq 5$ is due to Bertrand [1, p. 129] with an incomplete proof that relied on "Bertrand's postulate" that there is a prime strictly between n and $2n - 2$ for $n \geq 4$. He checked there is such a prime for n up to 3 million and it was proved in general by Chebyshev several years later.

Corollary 2.9. *Let F be a field. If $f \in F[X_1, \dots, X_n]$ and $n \geq 5$, the number of different polynomials we get from f by permuting its variables is either 1, 2, or at least n.*

Proof. Letting S_n act on $F[X_1, \dots, X_n]$ by permutations of the variables, the polynomials we get by permuting the variables of f is the S_n-orbit of f. The size of this orbit is $[S_n : H]$, where $H = \mathrm{Stab}_f = \{\sigma \in S_n : \sigma f = f\}$. By Theorem 2.7, this index is either 1, 2, or at least n. □

Corollary 2.9 is not true when $n = 4$. Here is a counterexample.

Example 2.10. In $F[X_1, X_2, X_3, X_4]$, let $f = X_1X_2 + X_3X_4$. Its S_4-orbit has 3 values:

$$X_1X_2 + X_3X_4, \quad X_1X_3 + X_2X_4, \quad X_1X_4 + X_2X_3.$$

3. First proof

Our first proof of Theorem 1.1 is based on the one in [3, pp. 149–150].

We begin by showing A_5 is simple.

Theorem 3.1. *The group A_5 is simple.*

Proof. We want to show the only normal subgroups of A_5 are $\{(1)\}$ and A_5. This will be done in two ways.

Our first method involves counting the orders of the conjugacy classes. There are 5 conjugacy classes in A_5, with representatives and orders as indicated in the following table.

Rep.	(1)	(12345)	(21345)	(12)(34)	(123)
Order	1	12	12	15	20

If A_5 has a normal subgroup N, then N is a union of conjugacy classes – including $\{(1)\}$ – whose total size divides 60. However, no sum of the above numbers that includes 1 is a factor of 60 except for 1 and 60. Therefore N is trivial or A_5.

For the second proof, let $N \lhd A_5$ with $|N| > 1$. We will show N contains a 3-cycle. It follows that $N = A_n$ by Lemmas 2.1 and 2.2.

Pick $\sigma \in N$ with $\sigma \neq (1)$. The cycle structure of σ is $(abc), (ab)(cd)$, or $(abcde)$, where different letters represent different numbers. Since we want to show N contains a 3-cycle, we may suppose σ has the second or third cycle type. In the second case, N contains

$$((abe)(ab)(cd)(abe)^{-1})(ab)(cd) = (be)(cd)(ab)(cd) = (aeb).$$

In the third case, N contains

$$((abc)(abcde)(abc)^{-1})(abcde)^{-1} = (adebc)(aedcb) = (abd).$$

Therefore N contains a 3-cycle, so $N = A_5$. $\square$

Lemma 3.2. *When $n \geq 5$, each nontrivial σ in A_n has a conjugate $\sigma' \neq \sigma$ such that $\sigma(i) = \sigma'(i)$ for some i.*

For example, if $\sigma = (12345)$ in A_5 then $\sigma' = (345)\sigma(345)^{-1} = (12453)$ has the same value at $i = 1$ as σ does.

Proof. Let r be the longest length of a disjoint cycle in σ. By replacing σ with a conjugate permutation (which is also in A_n and has the effect of just relabeling the numbers from 1 to n when σ permutes them), we can assume the disjoint r-cycle in σ is $(12\ldots r)$ and then we can write

$$\sigma = (12\ldots r)\pi,$$

where $(12\ldots r)$ and π are disjoint.

If $r \geq 3$, let $\tau = (345)$ and $\sigma' = \tau\sigma\tau^{-1}$. Then $\sigma(1) = 2, \sigma'(1) = 2, \sigma(2) = 3$, and $\sigma'(2) = 4$. Thus $\sigma' \neq \sigma$ and both take the same value at 1.

If $r = 2$, then σ is a product of disjoint transpositions. If there are at least 3 disjoint transpositions involved, then $n \geq 6$ and we can write $\sigma = (12)(34)(56)(\ldots)$ after relabelling. Let $\tau = (12)(35)$ and $\sigma' = \tau\sigma\tau^{-1}$. Then $\sigma(1) = 2, \sigma'(1) = 2, \sigma(3) = 4$, and $\sigma'(3) = 6$. Again, we see $\sigma' \neq \sigma$ and σ and σ' have the same value at 1.

If $r = 2$ and σ is a product of 2 disjoint transpositions, write $\sigma = (12)(34)$ after relabelling. Let $\tau = (132)$ and $\sigma' = \tau\sigma\tau^{-1} = (13)(24)$. Then $\sigma' \neq \sigma$ and they both fix 5. $\square$

Now we prove Theorem 1.1.

Proof. We may suppose $n \geq 6$, by Theorem 3.1. For $1 \leq i \leq n$, let A_n act in the natural way on $\{1, 2, \dots, n\}$ and let $H_i \subset A_n$ be the subgroup fixing i, so $H_i \cong A_{n-1}$. By induction, each H_i is simple. Note each H_i contains a 3-cycle (build out of 3 numbers other than i).

Let $N \lhd A_n$ be a nontrivial normal subgroup. We want to show $N = A_n$. Pick $\sigma \in N$ with $\sigma \neq \{(1)\}$. By Lemma 3.2, there is a conjugate σ' of σ such that $\sigma' \neq \sigma$ and $\sigma(i) = \sigma'(i)$ for some i. Since N is normal in A_n, $\sigma' \in N$. Then $\sigma^{-1}\sigma'$ is a non-identity element of N that fixes i, so $N \cap H_i$ is a nontrivial subgroup of H_i. It is also a normal subgroup of H_i since $N \lhd A_n$. Since H_i is simple, $N \cap H_i = H_i$. Therefore $H_i \subset N$. Since H_i contains a 3-cycle, N contains a 3-cycle and we are done.

Alternatively, we can show $N = A_n$ when $N \cap H_i$ is nontrivial for some i as follows. As before, since $N \cap H_i$ is a nontrivial normal subgroup of H_i, $H_i \subset N$. Without referring to 3-cycles, we instead note that the different H_i's are conjugate subgroups of A_n: $\sigma H_i \sigma^{-1} = H_{\sigma(i)}$ for $\sigma \in A_n$ Since $N \lhd A_n$ and N contains H_i, N contains every $H_{\sigma(i)}$ for all $\sigma \in A_n$. Since $\sigma(i)$ can be an arbitrary element of A_n as σ varies in A_n, N contains every H_i. Every permutation of type $(2, 2)$ is in some H_i since $n \geq 5$, so N contains all permutations of type $(2, 2)$. Every permutation in A_n is a product of permutations of type (2,2), so $N \supset A_n$. Therefore $N = A_n$. □

4. Second proof

Our next proof is taken from [8, p. 108]. It does not use induction on n, but we do need to know A_6 is simple at the start.

Theorem 4.1. *The group A_6 is simple.*

Proof. We follow the first method of proof of Theorem 3.1. Here is the table of conjugacy classes in A_6.

Rep.	(1)	(123)	(123)(456)	(12)(34)	(12345)	(23456)	(1234)(56)
Order	1	40	40	45	72	72	90

A tedious check shows no sum of these orders, which includes 1, is a factor of $6!/2$ except for the sum of all the terms. Therefore the only nontrivial normal subgroup of A_6 is A_6. □

Now we prove the simplicity of A_n for larger n by reducing directly to the case of A_6.

Proof. Since A_5 and A_6 are known to be simple by Theorems 3.1 and 4.1, pick $n \geq 7$ and let $N \lhd A_n$ be a nontrivial subgroup. We will show N contains a 3-cycle.

Let σ be a non-identity element of N. It moves some number. By relabelling, we may suppose $\sigma(1) \neq 1$. Let $\tau = (ijk)$, where i, j, k are not 1 and $\sigma(1) \in \{i, j, k\}$. Then $\tau\sigma\tau^{-1}(1) = \tau(\sigma(1)) \neq \sigma(1)$, so $\tau\sigma\tau^{-1} \neq \sigma$. Let $\varphi = \tau\sigma\tau^{-1}\sigma^{-1}$, so $\varphi \neq (1)$. Writing

$$\varphi = (\tau\sigma\tau^{-1})\sigma^{-1},$$

we see $\varphi \in N$. Now write

$$\varphi = \tau(\sigma\tau^{-1}\sigma^{-1}),$$

Since τ^{-1} is a 3-cycle, $\sigma\tau^{-1}\sigma^{-1}$ is also a 3-cycle. Therefore φ is a product of two 3-cycles, so φ moves at most 6 numbers in $\{1, 2, \dots, n\}$. Let H be the copy of A_6 inside A_n corresponding to the even permutations of those 6 numbers (possibly augmented to 6 arbitrarily if in fact φ moves fewer numbers). Then $N \cap H$ is nontrivial (it contains φ) and it is a normal subgroup of H. Since $H \cong A_6$, which is simple, $N \cap H = H$. Thus $H \subset N$, so N contains a 3-cycle. □

5. Third proof

Our next proof is by induction, and uses conjugacy classes instead of Lemma 3.2. It is based on [11, §2.3].

Lemma 5.1. *If $n \geq 6$ then every nontrivial conjugacy class in S_n and A_n has at least n elements.*

The lower bound n in Lemma 5.1 is actually quite weak as n grows. But it shows that the size of each nontrivial conjugacy class in S_n and A_n grows with n.

Proof. For $n \geq 6$, pick $\sigma \in S_n$ with $\sigma \neq (1)$. We want to look at the conjugacy class of σ in S_n, and if $\sigma \in A_n$ we also want to look at the conjugacy class of σ in A_n, and our goal in both cases is to find at least n elements in the conjugacy class.

Case 1: The disjoint cycle decomposition of σ includes a cycle with length greater than 2. Without loss of generality, $\sigma = (123\dots)\dots$.

For $3 \leq k \leq n$, fix a choice of $\ell \notin \{1, 2, 3, k\}$ (which is possible since $n \geq 5$) and let $\alpha_k = (2k\ell)$ and $\beta_k = (3k\ell)$. Then $\alpha_k\sigma\alpha_k^{-1}$ has the effect $1 \to 1 \to 2 \to k$ and $\beta_k\sigma\beta_k^{-1}$ has the effect $1 \to 1 \to 2 \to 2$ and $2 \to 2 \to 3 \to k$. This tells us that the conjugates

$$\alpha_3\sigma\alpha_3^{-1}, \dots, \alpha_n\sigma\alpha_n^{-1}, \beta_3\sigma\beta_3^{-1}, \dots, \beta_n\sigma\beta_n^{-1}$$

are all different from each other: the conjugates by the α's have different effects on 1, the conjugates by the β's have different effects on 2, and a conjugate by an α is not a conjugate by a β since they have different effects on 1. Since these conjugates are different, the number of conjugates of σ is at least $2(n-2) > n$. Because α_k and β_k are 3-cycles, if $\sigma \in A_n$ then these conjugates are in the A_n-conjugacy class of σ.

Case 2: The disjoint cycle decomposition of σ only has cycles with length 1 or 2. Therefore without loss of generality σ is a transposition or a product of at least 2 disjoint transpositions.

If σ is a transposition, then its S_n-conjugacy class is the set of all transpositions (ij) where $1 \leq i < j \leq n$, and the number of these permutations is $\binom{n}{2} = \frac{n^2-n}{2}$, which is greater than n for $n \geq 6$.

If σ is a product of at least 2 disjoint transpositions, then without loss of generality $\sigma = (12)(34)\dots$, where the terms in $\dots$ don't involve 1, 2, 3, or 4.

For $5 \leq k \leq n$, let $\alpha_k = (12)(3k)$, $\beta_k = (13)(2k)$, and $\gamma_k = (1k)(23)$. Then $\alpha_k\sigma\alpha_k^{-1}$ has the effect

$$1 \to 2 \to 1 \to 2, \quad 2 \to 1 \to 2 \to 1, \quad k \to 3 \to 4 \to 4,$$

$\beta_k\sigma\beta_k^{-1}$ has the effect

$$1 \to 3 \to 4 \to 4, \quad 3 \to 1 \to 2 \to k, \quad k \to 2 \to 1 \to 3,$$

and $\gamma_k\sigma\gamma_k^{-1}$ has the effect

$$2 \to 3 \to 4 \to 4, \quad 3 \to 2 \to 1 \to k, \quad k \to 1 \to 2 \to 3.$$

The conjugates of σ by the α's are different from each other since they take different elements to 4, the conjugates of σ by the β's are different from each other since they take different elements to 3, and the conjugates of σ by the γ's are different from each other since they take different elements to 3. Conjugates of σ by an α and a β are different since they send 1 to different places, conjugates of σ by an α and a γ are different since they send 2 to different places, and conjugates of σ by a β and a γ are different since they send different elements to 4 (1 for the β's and 2 for the γ's). In total the number of conjugates of σ we have written down (which are all conjugates by 3-cycles, hence they are conjugates in A_n if $\sigma \in A_n$) is $3(n-4)$, and $3(n-4) \geq n$ if $n \geq 6$. □

Now we prove Theorem 1.1.

Proof. We argue by induction on n, the case $n = 5$ having already been settled by Theorem 3.1. Say $n \geq 6$. Let $N \lhd A_n$ with $N \neq \{(1)\}$. Since N is normal and nontrivial, it contains non-identity conjugacy classes in A_n. By Lemma 5.1, each non-identity conjugacy class in A_n has size at least n when $n \geq 6$. Therefore, by counting the trivial conjugacy class and a nontrivial conjugacy class in N, we see $|N| \geq n+1$.

Using a wholly different argument, we now show that $|N| \leq n$ if $N \neq A_n$, which will be a contradiction. Pick $1 \leq i \leq n$. Let $H_i \subset A_n$ be the subgroup fixing i, so $H_i \cong A_{n-1}$. In particular, H_i is a simple group by induction. Notice each H_i contains a 3-cycle.

The intersection $N \cap H_i$ is a normal subgroup of H_i, so simplicity of H_i implies $N \cap H_i$ is either $\{(1)\}$ or H_i. If $N \cap H_i = H_i$ for some i, then $H_i \subset N$. Since H_i contains a 3-cycle, N does as well, so $N = A_n$ by Lemmas 2.1 and 2.2. (This part resembles part of our first proof of simplicity of A_n, but we will now use Lemma 5.1 instead of Lemma 3.2 to show the possibility that $N \cap H_i = \{(1)\}$ for all i is absurd.)

Suppose $N \neq A_n$. Then, by the previous paragraph, $N \cap H_i = \{(1)\}$ for all i. Therefore each $\sigma \neq (1)$ in N acts on $\{1, 2, \ldots, n\}$ without fixed points (otherwise σ would be a non-identity element in some $N \cap H_i$). That implies each $\sigma \neq (1)$ in N is completely determined by the value $\sigma(1)$: if $\tau \neq (1)$ is in N and $\sigma(1) = \tau(1)$, then $\sigma\tau^{-1} \in N$ fixes 1, so $\sigma\tau^{-1}$ is the identity, so $\sigma = \tau$.

There are only $n-1$ possible values for $\sigma(1) \in \{2, 3, \ldots, n\}$, so $N - \{(1)\}$ has size at most $n-1$, hence $|N| \leq n$. We already saw from Lemma 5.1 that $|N| \geq n+1$, so we have a contradiction. □

6. Fourth proof

Our next proof, based on [4, p. 50], is very computational.

Proof. Let $N \lhd A_n$ be a nontrivial normal subgroup. We will show N contains a 3-cycle.

Pick $\sigma \in N$, $\sigma \neq (1)$. Write

$$\sigma = \pi_1 \pi_2 \cdots \pi_k,$$

where the π_j's are disjoint cycles. In particular, they *commute*, so we can relabel them at our convenience. Eliminate all 1-cycles from the product.

<u>Case 1</u>: Some π_i has length at least 4. Relabelling, we can write

$$\pi_1 = (12 \cdots r)$$

with $r \geq 4$. Let $\varphi = (123)$. Then $\varphi\sigma\varphi^{-1} \in N$ and

$$\begin{aligned} \varphi\sigma\varphi^{-1} &= \varphi\pi_1\varphi^{-1}\pi_2 \cdots \pi_k \\ &= \varphi\pi_1\varphi^{-1}\pi_1^{-1}\sigma \\ &= (123)(123 \cdots r)(132)(r \cdots 21)\sigma \\ &= (124)\sigma, \end{aligned}$$

so $(124) = \varphi\sigma\varphi^{-1}\sigma^{-1} \in N$.

Case 2: Each π_i has length ≤ 3, and at least two have length 3 (so $n \geq 6$). Without loss of generality, $\pi_1 = (123)$ and $\pi_2 = (456)$. Let $\varphi = (124)$. Then

$$\begin{aligned} \varphi\sigma\varphi^{-1} &= \varphi\pi_1\pi_2\varphi^{-1}\pi_3\cdots\pi_k \\ &= \varphi\pi_1\pi_2\varphi^{-1}\pi_2^{-1}\pi_1^{-1}\sigma \\ &= (124)(123)(456)(142)(465)(132)\sigma \\ &= (12534)\sigma, \end{aligned}$$

so $\varphi\sigma\varphi^{-1}\sigma^{-1} = (12534) \in N$. Now run through Case 1 with this 5-cycle to find a 3-cycle in N.

Case 3: Exactly one π_i has length 3, and the rest have length ≤ 2. Without loss of generality, $\pi_1 = (123)$ and the other π_i's are 2-cycles. Then $\sigma^2 = \pi_1^2$ is in N, and $\pi_1^2 = (132)$.

Case 4: All π_i's are 2-cycles, so necessarily $k > 1$. Write $\pi_1 = (12)$ and $\pi_2 = (34)$. Let $\varphi = (123)$. Then

$$\begin{aligned} \varphi\sigma\varphi^{-1} &= \varphi\pi_1\pi_2\varphi^{-1}\pi_3\cdots\pi_k \\ &= \varphi\pi_1\pi_2\varphi^{-1}\pi_2^{-1}\pi_1^{-1}\sigma \\ &= (123)(12)(34)(132)(34)(12)\sigma \\ &= (13)(24)\sigma, \end{aligned}$$

so

$$\varphi\sigma\varphi^{-1}\sigma^{-1} = (13)(24) \in N.$$

Let $\psi = (135)$. Then

$$\begin{aligned} (13)(24)\psi(13)(24)\psi^{-1} &= (13)(24)(135)(13)(24)(153) \\ &= (13)(135)(13)(153) \\ &= (135), \end{aligned}$$

so N contains a 3-cycle. $\square$

7. Fifth proof

Our final proof is taken from [10, p. 295].

Let $N \lhd A_n$ with N not $\{(1)\}$ or A_n. We will study N as a subgroup of S_n. By Lemma 2.4, N is not a normal subgroup of S_n. This means the normalizer of N inside S_n is a proper subgroup, which contains A_n, so

$$A_n = \mathrm{N}_{S_n}(N). \tag{7.1}$$

For a transposition τ in S_n, $\tau \notin \mathrm{N}_{S_n}(N)$ by (7.1), so $\tau N\tau^{-1} \neq N$. Since $N \lhd A_n$ and $\tau N \tau^{-1}$ is a subgroup of A_n, the product set $N \cdot \tau N\tau^{-1}$ is a subgroup of A_n. We have the chain of inclusions

$$N \cap \tau N\tau^{-1} \subset N \subset N \cdot \tau N \tau^{-1} \subset A_n,$$

where the first and second are strict.

We will now show, for each transposition τ in S_n, that

$$N \cap \tau N\tau^{-1} \lhd S_n, \quad N \cdot \tau N\tau^{-1} \lhd S_n. \tag{7.2}$$

The proof of (7.2) is a bit tedious , so first let's see why (7.2) leads to a contradiction.

It follows from (7.2) and Lemma 2.4 that

$$N \cap \tau N\tau^{-1} = \{(1)\}, \quad N \cdot \tau N\tau^{-1} = A_n \tag{7.3}$$

for all transpositions τ in S_n. By (7.3), $|A_n| = |N| \cdot |\tau N \tau^{-1}| = |N|^2$, so $n! = 2|N|^2$. This tells us $|N|$ must be even, so N has an element, say σ, of order 2. Then σ is a product of disjoint 2-cycles. There is a transposition ρ in S_n that commutes with σ: just take for ρ one of the transpositions in the disjoint cycle decomposition of σ. Then

$$\sigma = \rho\sigma\rho^{-1} \in N \cap \rho N \rho^{-1}.$$

From (7.3), using ρ for the arbitrary τ there, $N \cap \rho N \rho^{-1}$ is trivial, so we have a contradiction. Another way of reaching a contradiction from the equation $n! = 2|N|^2$ uses Bertrand's postulate (Remark 2.8), which implies $n!/2$ can't be a perfect square since it is divisible once by a prime between $n!/4$ and $n!/2$.

It remains to check the two conditions in (7.2). In both cases, we show the subgroups are normalized by A_n and by τ, so the normalizer contains $\langle A_n, \tau\rangle = S_n$.

First consider $N \cap \tau N \tau^{-1}$. It is clearly normalized by τ. Now pick $\pi \in A_n$. Then $\pi N \pi^{-1} = N$ since $N \lhd A_n$, and

$$\pi(\tau N \tau^{-1})\pi^{-1} = \tau(\tau^{-1}\pi\tau)N(\tau^{-1}\pi^{-1}\tau)\tau^{-1} = \tau N \tau^{-1} \tag{7.4}$$

since $\tau^{-1}\pi\tau \in A_n$. Therefore

$$\pi(N \cap \tau N \tau^{-1})\pi^{-1} = \pi N \pi^{-1} \cap \pi\tau N \tau^{-1}\pi^{-1} = N \cap \tau N \tau^{-1},$$

so A_n normalizes $N \cap \tau N \tau^{-1}$.

Now we look at $N \cdot \tau N \tau^{-1}$. Pick an element of this product, say

$$\sigma = \sigma_1 \tau \sigma_2 \tau^{-1},$$

where $\sigma_1, \sigma_2 \in N$. Then, since $N \lhd A_n$,

$$\tau\sigma\tau^{-1} = \tau\sigma_1\tau\sigma_2\tau^{-2} = \tau\sigma_1\tau\sigma_2 \in \tau N \tau^{-1} \cdot N = N \cdot \tau N \tau^{-1},$$

which shows τ normalizes $N \cdot \tau N \tau^{-1}$.

Now pick $\pi \in A_n$. To see π normalizes $N \cdot \tau N \tau^{-1}$, pick σ as before. Then

$$\pi\sigma\pi^{-1} = \pi\sigma_1\pi^{-1} \cdot \pi(\tau\sigma_2\tau^{-1})\pi^{-1}.$$

The first factor $\pi\sigma_1\pi^{-1}$ is in N since $N \lhd A_n$. The second factor is in $\pi\tau N \tau^{-1}\pi^{-1}$, which equals $\tau N \tau^{-1}$ by (7.4).

8. Concluding Remarks

The standard counterexample to the converse of Lagrange's theorem is A_4: it has order 12 but no subgroup of index 2. For $n \geq 5$, the groups A_n also have no subgroup of index 2, since each index-2 subgroup of a group would be normal and A_n is simple.

In fact, something stronger is true.

Corollary 8.1. *For $n \geq 5$, each proper subgroup of A_n has index at least n.*

This is an analogue of Theorem 2.7, but its proof is more sophisticated.

Proof. Let H be a proper subgroup of A_n, with index $m > 1$. Consider the left multiplication action of A_n on A_n/H. This gives a group homomorphism

$$\varphi\colon A_n \to \mathrm{Sym}(A_n/H) \cong S_m.$$

Let K be the kernel of φ, so $K \subset H$ (why?) and $K \lhd A_n$. By *simplicity of A_n*, K is trivial. Therefore A_n injects into S_m, so $(n!/2) \mid m!$, which implies $n \leq m$. □

The lower bound of n is sharp since $[A_n : A_{n-1}] = n$. Corollary 8.1 is false for $n = 4$: A_4 has a subgroup of index 3.

Remark 8.2. What the proof of Corollary 8.1 shows more generally is that if G is a finite simple group and H is a subgroup with index $m > 1$, then there is an embedding of G into S_m, so $|G| \mid m!$. With G fixed, this divisibility relation puts a lower bound on the index of a proper subgroup of G.

A reader who wants to read more proofs that A_n is simple for $n \geq 5$ can see [5, pp. 247-248] or [7, pp. 32–33] for another way of showing a nontrivial normal subgroup contains a 3-cycle, or see [2, §1.7] or [9, pp. 295–296] for a proof based on the theory of highly transitive permutation groups.

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