

A SEPARABLE EXTENSION WITH INSEPARABLE RESIDUE FIELD

KEITH CONRAD

1. INTRODUCTION

Let K be a field complete with respect to a non-archimedean absolute value and have valuation ring \mathcal{O}_K and residue field $k = \mathcal{O}_K/\mathfrak{m}_K$. We will give examples of such K and a finite separable extension in which the residue field extension over k is inseparable.

2. THE EXAMPLES

Let F be an imperfect field of characteristic p , so F^p is a proper subset of F . Pick $a \in F - F^p$. (Example: $F = \mathbf{F}_p(y)$ and $a = y$.) It is a basic result in field theory that a not being a p th power in F makes $T^p - a$ irreducible in $F[T]$.

Example 2.1. Set $K = F((x))$, equipped with the x -adic valuation, so $\mathcal{O}_K = F[[x]]$, $\mathfrak{m}_K = (x)$, and $k = \mathcal{O}_K/\mathfrak{m}_K \cong F$. Set

$$f(T) = T^p - xT - a \in \mathcal{O}_K[T].$$

Reducing coefficients modulo x , $\bar{f}(T) = T^p - a$ is irreducible in $F[T]$ (because $a \notin F^p$). By Gauss's lemma, $f(T)$ is irreducible over K . Since $f'(T) = -x$ is a nonzero constant in K , $(f(T), f'(T)) = 1$. Thus $f(T)$ is separable.

Let $L = K(\alpha)$ where α is a root of $f(T)$, so L/K is separable and $[L : K] = p$. In the residue field $\ell := \mathcal{O}_L/\mathfrak{m}_L$, which is an extension of $k \cong F$, the element $\bar{\alpha}$ is a root of $\bar{f}(T) = T^p - a$. Since $\bar{f}(T)$ is irreducible in $F[T]$, $[\ell : k] \geq p$. It is always the case that the residue field degree does not exceed the field degree, so $[\ell : k] \leq [L : K] = p$. Thus $[\ell : k] = p$, so $\ell = k(\bar{\alpha})$, which is (purely) inseparable over k .

Let's show the field extension L/K is not Galois when $p > 2$. The roots of $f(T)$ are $\{\alpha + ct : c \in \mathbf{F}_p\}$ where $t^{p-1} = x$, so if $K(\alpha)/K$ were Galois then $t = (\alpha + t) - \alpha$ would be in $K(\alpha)$, but $t^{p-1} = x \Rightarrow [K(t) : K] = p - 1$, so t and α have relatively prime degrees over K and thus (since $p - 1 > 1$) $t \notin K(\alpha)$.

Remark 2.2. When $a \in F - F^p$, $T^{p^r} - a$ is irreducible for all $r \geq 1$, so for a root α_r of $T^{p^r} - xT - a$, $K(\alpha_r)$ is a separable extension of K with degree p^r and the residue field of $K(\alpha_r)$ is inseparable over k . The extension $K(\alpha_r)/K$ is not Galois when $p^r > 2$.

Example 2.3. We can modify the construction in Example 2.1 to get a *Galois* extension of K with an inseparable residue field extension. What was important about using $T^p - xT - a$ in Example 2.1 is that its T -coefficient is divisible by x and is not zero in \mathcal{O}_K . Set

$$f(T) = T^p - x^{p-1}T - a$$

As in Example 2.1, $f(T)$ is irreducible and separable in $K[T]$ since $\bar{f}(T) = T^p - a$ is irreducible in $F[T]$ and $f'(T) = -x^{p-1} \in K^\times$, so $(f(T), f'(T)) = 1$ in $K[T]$.

Let $L = K(\beta)$ where β is a root of $f(T)$, so L is separable over K of degree p . By the same reasoning as in Example 2.1, the residue field extension is $k(\bar{\beta})/k$, which is (purely)

inseparable of degree p . The roots of $f(T)$ are $\{\beta + cx : c \in \mathbf{F}_p\}$ since $f(\beta + cx) = (\beta^p + c^p x^p) - x^{p-1}(\beta + cx) - a = f(\beta) + (c^p - c)x^p = 0 + 0 = 0$. Thus L/K is Galois. In this step it's important that the coefficient of T is $-x^{p-1}$.

Replacing T by xT in $f(T)$, L can also be viewed as the splitting field of $T^p - T - a/x^p$ over K .