A SEPARABLE EXTENSION WITH INSEPARABLE RESIDUE FIELD

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Let K be a field complete with respect to a non-archimedean valuation, having valuation ring $\mathcal{O}_K$ and residue field $k = \mathcal{O}_K/\mathfrak{m}_K$. We describe here examples of such K with a finite extension that is separable over K and the residue field extension over k is inseparable.

Let F be an imperfect field of characteristic p, so F^p is a proper subset of F. Pick $a \in F - F^p$. (Example: $F = \mathbf{F}_p(y)$ and $a = y$.) It is a basic result in field theory that a not being a pth power in F makes $T^p - a$ irreducible in $F[T]$.

Example 1. Set $K = F((x))$, equipped with the x-adic valuation, so $\mathcal{O}_K = F[[x]]$, $\mathfrak{m}_K = (x)$, and $k = \mathcal{O}_K/\mathfrak{m}_K \cong F$. Set

$$f(T) = T^p - xT - a \in \mathcal{O}_K[T].$$

Reducing coefficients modulo x, $\overline{f}(T) = T^p - a$ is irreducible in $F[T]$ (because $a \notin F^p$) . By Gauss's lemma, $f(T)$ is irreducible over K. Since $f'(T) = -x$ is a nonzero constant in K, $(f(T), f'(T)) = 1$. Thus $f(T)$ is separable.

Let $L = K(\alpha)$ where α is a root of $f(T)$, so L/K is separable and $[L : K] = p$. In the residue field $\ell := \mathcal{O}_L/\mathfrak{m}_L$, which is an extension of $k \cong F$, the element $\overline{\alpha}$ is a root of $\overline{f}(T) = T^p - a$. Since $\overline{f}(T)$ is irreducible in $F[T]$, $[\ell : k] \geq p$. It is always the case that the residue field degree does not exceed the field degree, so $[\ell : k] \leq [L : K] = p$. Thus $[\ell : k] = p$, so $\ell = k(\overline{\alpha})$, which is (purely) inseparable over k.

Let's show the field extension L/K is not Galois when $p > 2$. The roots of $f(T)$ are $\{\alpha + ct : c \in \mathbf{F}_p\}$ where $t^{p-1} = x$, so if $K(\alpha)/K$ *were* Galois then $t = (\alpha + t) - \alpha$ would be in $K(\alpha)$, but $t^{p-1} = x \Rightarrow [K(t) : K] = p - 1$, so t and α have relatively prime degrees over K and thus (since $p - 1 > 1$) $t \notin K(\alpha)$.

Remark 2. When $a \in F - F^p$, $T^{p^r} - a$ is irreducible for all $r \geq 1$, so a root of $f(T) = T^{p^r} - xT - a$ over K generates a separable extension of K with degree p^r that has an inseparable residue field extension. It is not Galois when $p^r > 2$.

Example 3. We can modify the construction in Example 1 to get a *Galois* extension of K with an inseparable residue field extension. What was important about using $T^p - xT - a$ in Example 1 is that its T-coefficient is divisible by x and is not zero in $\mathcal{O}_K$. Set

$$f(T) = T^p - x^{p-1}T - a$$

As in Example 1, $f(T)$ is irreducible and separable in $K[T]$ since $\overline{f}(T) = T^p - a$ is irreducible in $F[T]$ and $f'(T) = -x^{p-1}$ is a nonzero constant in K, so $(f(T), f'(T)) = 1$ in $K[T]$.

Let $L = K(\beta)$ where β is a root of $f(T)$, so L is separable over K of degree p. By the same reasoning as in Example 1, the residue field extension is $k(\overline{\beta})/k$, which is (purely) inseparable of degree p. The extension L/K is Galois since the roots of $f(T)$ are $\{\beta + cx : c \in \mathbf{F}_p\}$ (this is where it's important that the coefficient of T is $-x^{p-1}$).

Replacing T by xT in $f(T)$, the Galois extension $K(\beta)/K$ with an inseparable residue field extension is also the splitting field over K of $T^p - T - a/x^p$.