THE p-ADIC EXPANSION OF RATIONAL NUMBERS

KEITH CONRAD

1. Introduction

In the positive real numbers, the decimal expansion of every positive rational number is eventually periodic[1] (*e.g.*, $21/55 = .3\overline{81} = .3818181\ldots$) and, conversely, every eventually periodic decimal expansion is a positive rational number. We will prove the set of all rational numbers can be characterized among the p-adic numbers a similar way: they are the p-adic numbers with eventually periodic p-adic expansions.

Example 1.1. In $\mathbf{Q}_3$

$$\frac{2}{5} = 1\overline{1210} = 1121012101210\ldots.$$

where the initial one-digit block "1" is followed by the repeating block 1210. Let's check this is correct:

$$\begin{aligned}
1\overline{1210} &= 1121012101210\ldots \\
&= 1 + 3(121012101210\ldots) \\
&= 1 + 3(1 + 2\cdot 3 + 3^2)(1 + 3^4 + 3^8 + 3^{12} + \cdots) \\
&= 1 + 3(16)\sum_{k\geq 0} 3^{4k} \\
&= 1 + \frac{48}{1-3^4} \\
&= 1 - \frac{48}{80} \\
&= \frac{32}{80} \\
&= \frac{2}{5}.
\end{aligned}$$

As above, throughout this note we will use the convention of writing p-adic expansions with the lowest-order terms *on the left*, in the same way power series are written $(a_0 + a_1x + a_2x^2 + \cdots)$. For example, in $\mathbf{Q}_p$ we write

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$$

rather than $-1 = \cdots + (p-1)p^2 + (p-1)p + (p-1)$. When writing positive integers in base p, we will write them with lowest-order terms *on the right* in order to match the way positive integers are written in base 10, and we'll include a subscript for the base. For example, 58 in base 3 is 2011_3 since $58 = 2\cdot 3^3 + 0\cdot 3^2 + 1\cdot 3 + 1$, but the 3-adic expansion of 58 is written in *reverse order* as 1102 and that means $1 + 1\cdot 3 + 0\cdot 3^2 + 2\cdot 3^3$.

[1] This characterization of $\mathbf{Q}_{>0}$ inside $\mathbf{R}_{>0}$ is not affected by some numbers having more than one decimal expansion, such as $.5 = .49999\ldots$, which are both eventually periodic: eventually all 0 or eventually all 9.

Multiplying and dividing a p-adic number by powers of p shifts the digits, but does not affect the property of having an eventually periodic p-adic expansion. Therefore it suffices to focus for the most part on numbers with p-adic absolute value 1, which are p-adic expansions of the form $c_0 + c_1 p + c_2 p^2 + \cdots$ where $0 \leq c_i \leq p-1$ and $c_0 \neq 0$.

2. Purely periodic expansions

As a warm-up, let's describe p-adic numbers with purely periodic p-adic expansions.

Theorem 2.1. *A rational number with p-adic absolute value 1 has a purely periodic p-adic expansion if and only if it lies in the real interval $[-1, 0)$.*

Proof. A purely periodic p-adic expansion having p-adic absolute value 1 with a repeating block of k digits looks like $\overline{n_0 n_1 \ldots n_{k-1}}$, where $0 \leq n_i \leq p-1$ and $n_0 \neq 0$. We can evaluate this as a fraction by summing geometric series in $\mathbf{Z}_p$:

$$\begin{aligned} \overline{n_0 n_1 \ldots n_{k-1}} &= 1(n_0 n_1 \ldots n_{k-1}) + p^k (n_0 n_1 \ldots n_{k-1}) + p^{2k}(n_0 n_1 \ldots n_{k-1}) + \cdots \\ &= (n_0 n_1 \ldots n_{k-1})(1 + p^k + p^{2k} + \cdots) \\ &= \frac{n_0 n_1 \ldots n_{k-1}}{1 - p^k}. \end{aligned} \tag{2.1}$$

The p-adic expansion in the numerator of (2.1), which is the base p number $(n_{k-1} \cdots n_1 n_0)_p$ with digits in reverse order, is an integer between 1 and $p^k - 1$ (it is not 0 since $n_0 \neq 0$), and we are dividing it by $1 - p^k = -(p^k - 1)$, so this purely periodic expansion is a rational number lying in the interval $[-1, 0)$.

Conversely, let r be a rational number with p-adic absolute value 1 that lies in $[-1, 0)$. We will show r can be written in the form (2.1), and then the calculations that led to (2.1) can be read in reverse to see r has a purely periodic p-adic expansion.

Since $|r|_p = 1$ and $r < 0$ we can write $r = a/b$ with numerator $a < 0$ and denominator $b \geq 1$ that are both not divisible by p. Since p and b are relatively prime, from elementary number theory we have $p^k \equiv 1 \bmod b$ for some $k \geq 1$. Thus $p^k = 1 + bb'$ for some positive integer b', so

$$r = \frac{a}{b} = \frac{ab'}{bb'} = \frac{-ab'}{1 - p^k}.$$

Set $N = -ab'$. Since $a < 0$, $N \in \mathbf{Z}^+$. From $-1 \leq r < 0$ we get $-1 \leq N/(1 - p^k) < 0$, so $0 < N \leq p^k - 1$. Thus N in base p has at most k digits: $N = n_0 + n_1 p + \cdots + n_{k-1} p^{k-1}$ where the digits n_i are between 0 and $p-1$. Hence r has the form (2.1). Since a and b' are not divisible by p, $|N|_p = 1$ so $n_0 \neq 0$. □

Remark 2.2. This theorem is not saying rationals in $[-1, 0)$ have purely periodic p-adic expansions. It says rationals in $[-1, 0)$ *with p-adic absolute value* 1 have purely periodic expansions.

Example 2.3. Let's work out the 3-adic expansion of $-5/11$, which is in $[-1, 0)$ with 3-adic absolute value 1. The least[2] $k \geq 1$ making $3^k \equiv 1 \bmod 11$ is $k = 5$, with $3^5 - 1 = 11 \cdot 22$, so

$$-\frac{5}{11} = -\frac{5 \cdot 22}{11 \cdot 22} = -\frac{110}{3^5 - 1} = \frac{110}{1 - 3^5}.$$

[2]It is not important to pick k minimal, but to do otherwise makes the periodic digit block appear longer, like writing $\overline{12}$ as $\overline{1212}$.

In base 3, $110 = 3^4 + 3^3 + 2 = 11002_3$. Its 3-adic expansion from *left to right* is 20011, so

$$-\frac{5}{11} = \frac{11002_3}{1-3^5} = \frac{20011}{1-3^5} = \overline{20011} = 2001120011\ldots.$$

As a check that this calculation is correct, add up the terms in the 3-adic expansion and get back $-5/11$:

$$\begin{aligned} 2001120011\ldots &= 2\sum_{i\geq 0} 3^{5i} + 3^3\sum_{i\geq 0} 3^{5i} + 3^4\sum_{i\geq 0} 3^{5i} \\ &= \frac{2}{1-3^5} + \frac{27}{1-3^5} + \frac{81}{1-3^5} \\ &= \frac{2+27+81}{-242} \\ &= -\frac{110}{242} \\ &= -\frac{11\cdot 10}{11\cdot 22} \\ &= -\frac{5}{11}. \end{aligned}$$

We can get the p-adic expansion of a rational number in the real interval $(0,1)$ having p-adic absolute value 1 by using Theorem 2.1 to get the expansion of its negative and then negating the result. Recall the simple rule for negating a nonzero p-adic expansion: if $x = c_d p^d + c_{d+1}p^{d+1} + \cdots + c_i p^i + \cdots$ where the c_i are digits and $c_d \neq 0$, then

$$-x = (p-c_d)p^d + (p-1-c_{d+1})p^{d+1} + \cdots + (p-1-c_i)p^i + \cdots. \tag{2.2}$$

In the expansion of $-x$, note the first digit is affected differently from the rest: $p - c_d$ compared to $p-1-c_i$ for $i > d$.

Example 2.4. Let's derive the 3-adic expansion of $2/5$, which was pulled out of nowhere in Example 1.1. We will use the proof of Theorem 2.1 to find the expansion of $-2/5$ and then negate the result.

To make $3^k \equiv 1 \bmod 5$ we can use $k = 4$. Then $3^k - 1 = 5\cdot 16$, so

$$-\frac{2}{5} = -\frac{2\cdot 16}{5\cdot 16} = \frac{32}{1-3^4}.$$

In base 3, $32 = 3^3 + 3 + 2 = 1012_3$, so

$$-\frac{2}{5} = \frac{1012_3}{1-3^4} = \frac{2101}{1-3^4} = \overline{2101} = 210121012101\ldots,$$

which is purely periodic. Negating and using (2.2) with $p = 3$, we get

$$\frac{2}{5} = -210121012101\ldots = 112101210121\ldots = 1\overline{1210},$$

which is eventually periodic rather than purely periodic.

3. Eventually periodic expansions

Theorem 3.1. *In $\mathbf{Q}_p$, the numbers with eventually periodic p-adic expansions are precisely the rational numbers.*

Proof. We begin by showing every eventually periodic p-adic expansion is rational. This will generalize the calculations at the start of the proof of Theorem 2.1. An eventually periodic p-adic expansion with absolute value 1 looks like

$$m_0m_1\cdots m_{j-1}\overline{n_0n_1\cdots n_{k-1}} = m_0m_1\cdots m_{j-1}n_0n_1\cdots n_{k-1}n_0n_1\cdots n_{k-1}\cdots, \tag{3.1}$$

a first block of j digits $m_0m_1\cdots m_{j-1}$ followed by a repeating block of k digits $n_0n_1\cdots n_{k-1}$. (If the expansion is purely periodic then the initial block can be taken as empty and set $j = 0$.) Write (3.1) in series form as

$$m_0 + \cdots + m_{j-1}p^{j-1} + (n_0p^j + \cdots + n_{k-1}p^{j+k-1}) + (n_0p^{j+k} + \cdots + n_{k-1}p^{j+2k-1}) + \cdots.$$

Using geometric series, we evaluate (3.1):

$$\begin{aligned} m_0\ldots m_{j-1}\overline{n_0\ldots n_{k-1}} &= m_0\ldots m_{j-1} + (n_0\ldots n_{k-1})(p^j + p^{j+k} + p^{j+2k} + \cdots) \\ &= m_0\ldots m_{j-1} + p^j(n_0\ldots n_{k-1})(1 + p^k + p^{2k} + \cdots) \\ &= m_0\ldots m_{j-1} + p^j\frac{n_0\ldots n_{k-1}}{1-p^k} \\ &= (m_{j-1}\ldots m_0)_p + p^j\frac{(n_{k-1}\ldots n_0)_p}{1-p^k}, \end{aligned}$$

which is a rational number. (This generalizes the calculations that led to (2.1), which is the special case $j = 0$.) Allowing multiplication or division by powers of p, we have shown all eventually periodic p-adic expansions are rational numbers.

To prove the converse, that every rational number r has an eventually periodic p-adic expansion, we will, perhaps surprisingly, focus on *negative* r. The p-adic expansion of a positive rational number can be obtained from its negative by negating with (2.2), which clearly shows the negation of an eventually periodic p-adic expansion is eventually periodic. (If $r \in \mathbf{Z}^+$ there's really no need to negate first: the base p expansion of r is its p-adic expansion.)

Case 1: $r \in \mathbf{Z}$ with $r < 0$. Write $r = -R$ with $R \in \mathbf{Z}^+$. There is a $j \geq 1$ such that $R < p^j$. Then

$$r = -R = (p^j - R) - p^j.$$

Since $p^j - R$ is an integer in $\{1, \ldots, p^j - 1\}$ we can write it in base p as $c_0 + \cdots + c_{j-1}p^{j-1}$. Then

$$r = (p^j - R) - p^j = \sum_{i=0}^{j-1} c_ip^i + \sum_{i\geq j}(p-1)p^i,$$

which is eventually periodic since its digits eventually all equal $p - 1$.

Case 2: $r \in \mathbf{Q} \cap \mathbf{Z}_p^\times \cap (-1, 0)$. The p-adic expansion of r is purely periodic by Theorem 2.1, and the proof of that theorem shows how to obtain the expansion.

Case 3: $r \in \mathbf{Q}\cap\mathbf{Z}_p\cap(-1, 0)$. Write $r = p^nu$ with $u \in \mathbf{Z}_p^\times$. Then $u = r/p^n$ is rational, of p-adic absolute value 1, and is in the interval $(-1/p^n, 0) \subset (-1, 0)$, so u has a purely periodic p-adic expanion by Case 2. Therefore $r = p^nu$ has the same purely periodic expansion except for starting n positions further to the right.

Case 4: $r \in \mathbf{Q} \cap \mathbf{Z}_p$, $r \notin \mathbf{Z}$, and $r < -1$. The number r lies strictly between two negative integers: $-(N + 1) < r < -N$ for some positive integer N, so $-1 < r + N < 0$. Since $r + N \in \mathbf{Z}_p$, by Case 3 the p-adic expansion of $r + N$ is purely periodic, although not

necessarily starting at the p^0-digit (since $r+N$ might not be in $\mathbf{Z}_p^\times$), so we can write

$$r + N = \sum_{i \geq 0} a_i p^i \tag{3.2}$$

where $a_i \in \{0, 1, \ldots, p-1\}$ and the a_i are purely periodic after a possible initial string of zero digits. Since $r+N$ is not a positive integer, the p-adic expansion (3.2) has infinitely many nonzero a_i. Thus the partial sums $a_0 + a_1 p + \cdots + a_{j-1}p^{j-1}$ become arbitrarily large in the usual sense as j grows, so there is a j such that

$$a_0 + a_1 p + \cdots + a_{j-1}p^{j-1} > N. \tag{3.3}$$

Let j be the smallest choice fitting this inequality, so $a_{j-1} \neq 0$. Then

$$r + N = (a_0 + a_1 p + \cdots + a_{j-1}p^{j-1}) + \sum_{i \geq j} a_i p^i$$

so

$$r = (a_0 + a_1 p + \cdots + a_{j-1}p^{j-1} - N) + \sum_{i \geq j} a_i p^i \tag{3.4}$$

and the difference $a_0 + a_1 p + \cdots + a_{j-1}p^{j-1} - N$ is a positive integer by (3.3) that is less than $(p-1) + (p-1)p + \cdots + (p-1)p^{j-1} = p^j - 1$, so we can write the difference in base p:

$$a_0 + a_1 p + \cdots + a_{j-1}p^{j-1} - N = a_0' + a_1' p + \cdots + a_{j-1}' p^{j-1}$$

with $0 \leq a_i' \leq p-1$, so (3.4) becomes

$$r = (a_0' + a_1' p + \cdots + a_{j-1}' p^{j-1}) + \sum_{i \geq j} a_i p^i.$$

This is an eventually periodic p-adic expansion since the a_i for $i \geq j$ are eventually periodic.

Case 5: $r \in \mathbf{Q}$, $r \notin \mathbf{Z}_p$, $r < 0$. Since $p^e r \in \mathbf{Z}_p$ for large e, we can use a previous case on $p^e r$ and then divide by p^e. □

4. Examples

The proof of Theorem 3.1 gives an algorithm to compute the p-adic expansion of any rational number in $\mathbf{Z}_p$:

(1) Assume $r < 0$. (If $r > 0$, apply the rest of the algorithm to $-r$ and then negate with (2.2) to get the expansion for r.)
(2) If $r \in \mathbf{Z}_{<0}$ then write $r = -R$ and pick $j \geq 1$ such that $R < p^j$. Then $r = (p^j - R) - p^j = (p^j - R) + \sum_{i \geq j}(p-1)p^i$ and $p^j - R$ has a base p expansion not going beyond the p^{j-1}-digit.
(3) If $-1 < r < 0$ let $r = p^n u$ with $u \in \mathbf{Z}_p^\times$. Then $u \in (-1, 0)$ and the p-adic expanion of u is purely periodic using the proof of Theorem 2.1. Multiplying it by p^n gives the (purely periodic) p-adic expansion of r.
(4) If $-(N+1) < r < -N$ for an integer $N \geq 1$ then $-1 < r + N < 0$, so the expansion of $r+N$ is obtained by the previous step, say $r + N = \sum_{i \geq 0} a_i p^i$. Pick the first truncation $a_0 + a_1 p + \cdots + a_{j-1}p^{j-1}$ in this expansion that exceeds N, so $r = (\sum_{i=0}^{j-1} a_i p^i - N) + \sum_{i \geq j} a_i p^i$. The difference in parentheses is a positive integer and its base p expansion has the form $\sum_{i=0}^{j-1} a_i' p^i$, so $r = \sum_{i=0}^{j-1} a_i' p^i + \sum_{i \geq j} a_i p^i$.

Example 4.1. Let's work out the p-adic expansion of $77/18$ in $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Q}_5$, and $\mathbf{Q}_7$.

Expansion of $77/18$ in $\mathbf{Q}_2$: Since $77/18 = (1/2)(77/9)$ and $|77/9|_2 = 1$, we will get the 2-adic expansion of $77/9$ and then divide through by 2. And since $77/9 > 0$, we will first get the 2-adic expansion of $-77/9$ and then negate what we find.

Let $r = -77/9$. Since $-9 < r < -8$, set $N = 8$. Since $-1 < r + 8 < 0$ and $r + 8 = -5/9 \in \mathbf{Z}_2^\times \cap (-1, 0)$ we will find the 2-adic expansion of $-5/9$ by Theorem 2.1. The least k making $2^k \equiv 1 \bmod 9$ is $k = 6$:

$$2^6 - 1 = 63 = 9 \cdot 7 \Longrightarrow -\frac{5}{9} = -\frac{5 \cdot 7}{63} = \frac{35}{1 - 2^6}.$$

In base 2, $35 = 2^5 + 2 + 1 = 100011_2$, and its 2-adic expansion is $1 + 2 + 2^5 = 110001$, so

$$\frac{35}{1 - 2^6} = \frac{100011_2}{1 - 2^6} = \frac{110001}{1 - 2^6} = \overline{110001} = 110001110001110001\ldots$$

The first truncation of this that exceeds $N = 8$ is $110001 = 35$, so

$$r = -8 - \frac{5}{9} = -8 + 110001 + 000000\overline{110001} = (35 - 8) + 000000\overline{110001}.$$

Since $35 - 8 = 27 = 2^4 + 2^3 + 2 + 1 = 11011_2$, which has 2-adic expansion $1 + 2 + 2^3 + 2^4 = 11011$ (it is palindromic, a coincidence), we get

$$r = -\frac{77}{9} = 11011 + 000000\overline{110001} = 110110\overline{110001}.$$

Thus

$$\frac{77}{9} = -110110\overline{110001} = 101001\overline{001110},$$

so

$$\frac{77}{18} = \frac{101001\overline{001110}}{2} = \frac{1}{2} + 01001\overline{001110}.$$

Let's check: in $\mathbf{Q}_2$,

$$\frac{1}{2} + 01001\overline{001110} = \frac{1}{2} + (2 + 16) + 2^5\frac{4 + 8 + 16}{1 - 2^6} = \frac{1}{2} + 18 + 32\frac{28}{1 - 64} = \frac{37}{2} - \frac{32 \cdot 4}{9} \overset{\checkmark}{=} \frac{77}{18}.$$

Expansion of $77/18$ in $\mathbf{Q}_3$: Since $77/18 = (1/9)(77/2)$, first we will figure out the 3-adic expansion of $77/2$ and then divide it by 9. Since $77/2 > 0$, first we will compute the 3-adic expansion of $-77/2$ and then negate.

Let $r = -77/2$, so $-39 < r < -38$. We have $r + 38 = -1/2$, which is easy to expand 3-adically:

$$-\frac{1}{2} = \frac{1}{1 - 3} = \overline{1} = 111\ldots$$

and the first truncation of this 3-adic expansion that exceeds 38 is $1111 = 40$, so

$$r = -38 - \frac{1}{2} = -38 + 1111 + 0000\overline{1} = (40 - 38) + 0000\overline{1} = 2000\overline{1}.$$

Therefore

$$\frac{77}{2} = -2000\overline{1} = 1222\overline{1}$$

so

$$\frac{77}{18} = \frac{1222\overline{1}}{9} = \frac{1}{9} + \frac{2}{3} + 22\overline{1}.$$

Let's check: in $\mathbf{Q}_3$,

$$\frac{1}{9}+\frac{2}{3}+22\overline{1}=\frac{1}{9}+\frac{2}{3}+(2+2\cdot 3)+\frac{9}{1-3}=\frac{7}{9}+8-\frac{9}{2}=\frac{14+18\cdot 8-81}{18}\overset{\checkmark}{=}\frac{77}{18}.$$

Expansion of $77/18$ in $\mathbf{Q}_5$: We'll get the expansion for $-77/18$ and then negate.

Let $r=-77/18$. Since $-5<r<-4$, set $N=4$. Then $-1<r+4<0$ and $r+4=-5/18=5(-1/18)=5u$ where $u=-1/18\in\mathbf{Z}_5^\times\cap(-1,0)$. We will get the 5-adic expansion of $-1/18$ using Theorem 2.1 and then multiply through by 5.

The least k making $5^k\equiv 1 \bmod 18$ is $k=6$:

$$5^6-1=15624=18\cdot 868\Longrightarrow -\frac{1}{18}=-\frac{868}{15624}=\frac{868}{1-5^6}.$$

In base 5, $868=5^4+5^3+4\cdot 5^2+3\cdot 5+3=11433_5$, whose 5-adic expansion is 33411, so

$$u=\frac{868}{1-5^6}=\frac{11433_5}{1-5^6}=\frac{33411}{1-5^6}=\overline{334110}=33411033411033411\ldots$$

Thus

$$-\frac{5}{18}=5u=\overline{033411}.$$

The first truncation of this that exceeds $N=4$ is 03, which is 15, so

$$r=-4-\frac{5}{18}=-4+03+00\overline{341103}=(15-4)+00\overline{341103}.$$

Since $15-4=11=2\cdot 5+1=21_5$, which has 5-adic expansion $1+2\cdot 5=12$, we have

$$r=-\frac{77}{18}=12+00\overline{341103}=12\overline{341103}.$$

Thus

$$\frac{77}{18}=-12\overline{341103}=42\overline{103341}.$$

Let's check: in $\mathbf{Q}_5$,

$$42\overline{103341}=4+2\cdot 5+5^2\frac{1+3\cdot 5^2+3\cdot 5^3+4\cdot 5^4+5^5}{1-5^6}=14+25\frac{6076}{1-5^6}=14-25\frac{7}{18}\overset{\checkmark}{=}\frac{77}{18}.$$

Expansion of $77/18$ in $\mathbf{Q}_7$: We'll get the expansion for $-11/18$ and then multiply by -7.

Let $r=-11/18$. It lies in $\mathbf{Z}_7^\times\cap(-1,0)$ so we can compute its 7-adic expansion from Theorem 2.1.

The least k making $7^k\equiv 1 \bmod 18$ is $k=3$:

$$7^3-1=342=18\cdot 19\Longrightarrow -\frac{11}{18}=-\frac{11\cdot 19}{342}=\frac{209}{1-7^3}.$$

In base 7, $209=4\cdot 7^2+7+6=416_7$, which has 7-adic expansion $6+7+4\cdot 7^2=614$, so

$$r=\frac{209}{1-7^3}=\frac{416_7}{1-7^3}=\frac{614}{1-7^3}=\overline{614}=614614614\ldots$$

Therefore

$$\frac{11}{18}=-614614614\ldots=152052052\ldots=1\overline{520}$$

so

$$\frac{77}{18}=7\left(\frac{11}{18}\right)=01\overline{520}.$$

Let's make our final check: in $\mathbf{Q}_7$,

$$01\overline{520} = 7 + 7^2\frac{5 + 2 \cdot 7}{1 - 7^3} = 7 - 49\frac{19}{342} = 7 - \frac{49}{18} \overset{\checkmark}{=} \frac{77}{18}.$$