THE p-ADIC GROWTH OF HARMONIC SUMS

KEITH CONRAD

The numbers

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

are called *harmonic sums*. Integrals can approximate these sums as real numbers, *e.g.*, H_n and $\int_1^n \mathrm{d}t/t = \log n$ differ by a bounded amount ($H_n - \int_1^n \mathrm{d}t/t$ is positive and montonically decreasing), so the H_n's slowly diverge in $\mathbf{R}$. We are interested here in the arithmetic behavior of harmonic sums, *i.e.*, their p-adic growth. This is definitely not an important topic, but it is interesting to compare what happens with this rational sequence across the real and different p-adic landscapes.

Here are some initial values:

n	H_n
1	1
2	3/2
3	11/6
4	25/12
5	137/60
6	49/20
7	363/140
8	761/280
9	7129/2520
10	7381/2520
11	83711/27720

TABLE 1. Harmonic sums

For $n > 1$, it appears that $H_n \notin \mathbf{Z}$. More precisely, the numerator of H_n is odd and the denominator of H_n is even. This suggests a strategy for proving H_n is not an integer.

Theorem 1 (Theisinger)**.** *For all $n \geq 2$, $H_n \notin \mathbf{Z}$.*

Proof. Let L be the least common multiple of $1, 2, \ldots, n$, so H_n can be written as a fraction with denominator L. For $1 \leq k \leq n$, write $L = ka_k$ with $a_k \in \mathbf{Z}^+$, so $1/k = a_k/L$. Then

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{\sum_{k=1}^{n} a_k}{L}.$$

Since $n \geq 2$, L is even. We will show $\sum_{k=1}^{n} a_k$ is odd, so the ratio is not integral.

Set 2^r to be the largest power of 2 up to n: $2^r \leq n < 2^{r+1}$. The only integer up to n divisible by 2^r is 2^r itself, since $2 \cdot 2^r > n$. Therefore $L = 2^r b$ where b is odd, so $2^r b = ka_k$ for $1 \leq k \leq n$. When $k = 2^r$ we see that $a_k = b$ is odd. When $k \neq 2^r$, k is not divisible by 2^r, so a_k must be even. Therefore in the numerator $\sum_{k=1}^{n} a_k$, one term (at $k = 2^r$) is odd and the rest are even, so the total sum is odd. □

Theisinger [4] did not prove Theorem 1 using anything p-adic: he used determinants and Bertrand's postulate, which says there is always a prime between n and $2n$ when $n \geq 2$.[1]

The proof above for Theorem 1 presents it as a 2-adic result: $H_n \notin \mathbf{Z}$ because $H_n \notin \mathbf{Z}_2$. Here is a second proof of Theorem 1 brings out 2-adic features more clearly.

Proof. Let $2^r \leq n < 2^{r+1}$, so $r \geq 1$ and the highest power of 2 that appears in some reciprocal in the sum defining H_n is 2^r. The only reciprocal in H_n with denominator divisible by 2^r is $1/2^r$. Indeed, every other such reciprocal would be $1/(2^r c)$ for odd $c > 1$, but if that term is in the sum then so is $1/(2^r \cdot 2) = 1/2^{r+1}$, which is false since $2^{r+1} > n$. Therefore $1/2^r$ has a more highly negative 2-adic valuation than every other term in H_n, so it is not cancelled out in the sum. This means $H_n \notin \mathbf{Z}_2$, so $H_n \notin \mathbf{Z}$. $\square$

This proof gives a formula for the 2-adic valuation of the harmonic sums: $\operatorname{ord}_2(H_n) = -r$ where $2^r \leq n < 2^{r+1}$. To see this formula in action, we rewrite the initial harmonic sums in Table 2 with the power of 2 in the denominators made explicit. The exponent jumps when n is a power of 2.

n	H_n
1	1
2	$3/2$
3	$11/2 \cdot 3$
4	$25/2^2 \cdot 3$
5	$137/2^2 \cdot 15$
6	$49/2^2 \cdot 5$
7	$363/2^2 \cdot 35$
8	$761/2^3 \cdot 35$
9	$7129/2^3 \cdot 315$
10	$7381/2^3 \cdot 2520$
11	$83711/2^3 \cdot 3465$

TABLE 2. Harmonic sums viewed 2-adically

Since $|H_n|_2 = 2^r$ if $2^r \leq n < 2^{r+1}$, $|H_n|_2 \to \infty$ as $n \to \infty$ (explicitly, $n/2 < |H_n|_2 \leq n$). Does $|H_n|_p \to \infty$ as $n \to \infty$ for odd primes p? We will address this later.

Theorem 2 (Kürschák)**.** *For $m \leq n-2$, $H_n - H_m \notin \mathbf{Z}_2$. In particular, $H_n - H_m \notin \mathbf{Z}$.*

Taking $m = 1$ and $n \geq 3$ recovers Theorem 1 for $n \geq 3$ since $H_1 \in \mathbf{Z}$. Theorem 2 is false if $m = n-1$ and n is odd, since then $H_n - H_m = 1/n \in \mathbf{Z}_2$.

Proof. Writing

$$H_n - H_m = \sum_{k=m+1}^{n} \frac{1}{k},$$

we will show there is a unique term in the sum with the most negative 2-adic valuation. Let $r = \max_{m<k\leq n} \operatorname{ord}_2(k)$. Since $n \geq m+2$, the sum $H_n - H_m$ has at least two terms in it, so some k is even and therefore $r \geq 1$.

We will show there is only one integer from m up to $n-1$ with 2-adic valuation r. Suppose there are two such numbers. Write them as $2^r c$ and $2^r d$ with odd $c < d$. Then $c+1$ is even and $2^r c < 2^r(c+1) < 2^r d$, so $1/(2^r(c+1))$ appears in $H_n - H_m$. But $\operatorname{ord}_2(2^r(c+1)) \geq r+1$ since c is odd. This contradicts the definition of r. Therefore there is only one term in $H_n - H_m$ with 2-adic valuation $-r$, so $\operatorname{ord}_2(H_n - H_m) = -r$. $\square$

[1] Theisinger referred to Bertrand's postulate as Chebyshev's theorem, since Chebyshev had proved it.

This type of 2-adic argument (in the setting of rational numbers, not 2-adic numbers) is due to Kürschák [3], so he should be credited with the 2-adic proof of Theorem 1.

We now turn to the p-adic behavior of H_n for $p \neq 2$. The material below is based on the notes "Harmonics and Primes" by Nicholas Rogers.

The harmonic sums $H_1, H_2, \ldots, H_{p-1}$ are all p-adically integral, since the denominators appearing in them are prime to p and thus are p-adic integers (even p-adic units). Since $1/p$ is outside the ring $\mathbf{Z}_p$, $H_p = H_{p-1} + 1/p$ is outside $\mathbf{Z}_p$: $\mathrm{ord}_p(H_p) = -1$. It need not be true (as it is for $p = 2$) that $H_n \notin \mathbf{Z}_p$ for all $n \geq p$. For example, H_3, H_4 and H_5 are outside $\mathbf{Z}_3$ (all with 3-adic valuation -1), but H_6, H_7, and H_8 are in $\mathbf{Z}_3$. Then $H_9, H_{10}, \ldots, H_{20}$ are outside of $\mathbf{Z}_3$. The harmonic sums return to $\mathbf{Z}_3$ for H_{21}, H_{22}, and H_{23}, then leave $\mathbf{Z}_3$ and come back in again for H_{66}, H_{67}, and H_{68}. Then the harmonic sums leave $\mathbf{Z}_3$ and never return. In fact, $|H_n|_3 \to \infty$ as $n \to \infty$. We will not discuss the proof of this, but the next theorem gives a simple connection between p-adic integrality and p-adic divergence of the harmonic sums.

Theorem 3. *For a prime p, the following conditions are equivalent:*

(1) $\{n \geq 1 : H_n \in \mathbf{Z}_p\}$ *is finite,*
(2) $|H_n|_p \to \infty$ *as* $n \to \infty$.

Proof. Trivially the second condition implies the first. To prove the (more interesting) reverse implication, pick $n \geq 1$ and write $n = pq + r$ with $0 \leq r \leq p-1$. Since $1/k \in \mathbf{Z}_p$ when $(p,k) = 1$, in the quotient group $\mathbf{Q}_p/\mathbf{Z}_p$ a harmonic sum is equal to the sum of its terms whose denominators are divisible by p, so

$$H_n \equiv \frac{1}{p} + \frac{1}{2p} + \cdots + \frac{1}{pq} = \frac{1}{p}H_q,$$

so $H_n - (1/p)H_q \in \mathbf{Z}_p$.

We are assuming that $|H_n|_p \leq 1$ only finitely many times, so there is some $N_0 \geq 1$ such that $|H_n|_p > 1$ for $n \geq N_0$. We will show this N_0 controls the p-adic divergence of the harmonic sums: for all $k \geq 0$,

$$n \geq p^k N_0 \Longrightarrow |H_n|_p > p^k.$$

For $k = 0$ this is true by the definition of N_0. Assuming it is true for k, suppose $n \geq p^{k+1}N_0$. Writing $n = pq + r$ with $0 \leq r \leq p-1$, from $pq + r \geq p^{k+1}N_0$ we must have $q \geq p^k N_0$, so $|H_q|_p > p^k$. Therefore $|(1/p)H_q|_p > p^{k+1}$. Since $H_n - (1/p)H_q \in \mathbf{Z}_p$,

$$\left|H_n - \frac{1}{p}H_q\right|_p \leq 1 < p^{k+1} = \left|\frac{1}{p}H_q\right|_p,$$

so by the nonarchimedean property $|H_n|_p = |(1/p)H_q|_p > p^{k+1}$. $\square$

The same reasoning shows $|H_n|_p \to \infty$ as $n \to \infty$ if and only if the set $\{n \geq 1 : H_n \in p\mathbf{Z}_p\}$ is finite. (Replace the strict inequalities $|H_n|_p > 1$ and $|H_n|_p > p^k$ in the proof with $|H_n|_p \geq 1$ and $|H_n|_p \geq p^k$.) This set, introduced in [2], is denoted

$$J(p) = \{n \geq 1 : H_n \in p\mathbf{Z}_p\}.$$

To say $n \notin J(p)$ means H_n has no p in its numerator. For instance, Theorem 1 says $J(2) = \emptyset$. In [2] it is proved that $J(3) = \{2, 7, 22\}$, $J(5) = \{4, 20, 24\}$, and

$$J(7) = \{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}.$$

The set $J(11)$ has 638 elements [1]. It is conjectured that $J(p)$ is finite for all p, which by Theorem 3 is equivalent to saying $|H_n|_p \to \infty$ as $n \to \infty$ for all p.

The rest of our discussion will focus on the particular sum H_{p-1}.

Theorem 4. *For each odd prime p, $H_{p-1} \in p\mathbf{Z}_p$. Therefore $J(p) \neq \emptyset$ for $p > 2$.*

Proof. In the sum

$$H_{p-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{p-1},$$

each term is a unit in $\mathbf{Z}_p$, so we can reduce modulo p and check that $H_{p-1} \equiv 0 \bmod p\mathbf{Z}_p$. The integers $\{1, 2, \ldots, p-1\}$ represent the units modulo p, so their inverses do as well. Thus, after replacing each $1/k$ with the integer from 1 to $p-1$ that is equal to it in $\mathbf{Z}_p/(p)$,

$$\begin{aligned} H_{p-1} &\equiv 1 + 2 + \cdots + p - 1 \bmod p \\ &\equiv \frac{p(p-1)}{2} \bmod p \\ &\equiv 0 \bmod p, \end{aligned}$$

where in the last step we needed $p \neq 2$. □

In Table 3, we factor the initial harmonic sums into primes, and can see in this data (for $p = 5, 7$, and 11) that not only is H_{p-1} divisible by p, but by p^2.

n	H_n
1	1
2	$3/2$
3	$11/2 \cdot 3$
4	$5^2/2^2 \cdot 3$
5	$137/2^2 \cdot 3 \cdot 5$
6	$7^2/2^2 \cdot 5$
7	$3 \cdot 11^2/2^2 \cdot 5 \cdot 7$
8	$761/2^3 \cdot 5 \cdot 7$
9	$7129/2^3 \cdot 3^2 \cdot 5 \cdot 7$
10	$11^2 \cdot 61/2^3 \cdot 3^2 \cdot 5 \cdot 7$
11	$97 \cdot 863/2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$

TABLE 3. Harmonic sums fully factored

Theorem 5 (Wolstenholme [5], 1862)**.** *For each prime $p \geq 5$, $H_{p-1} \in p^2\mathbf{Z}_p$.*

Proof. We collect terms in H_{p-1} that are equidistant from the middle of the sum:

$$\begin{aligned} H_{p-1} &= 1 + \frac{1}{2} + \cdots + \frac{1}{p-1} \\ &= \left(1 + \frac{1}{p-1}\right) + \left(\frac{1}{2} + \frac{1}{p-2}\right) + \cdots + \left(\frac{1}{(p-1)/2} + \frac{1}{(p+1)/2}\right) \\ &= \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k} + \frac{1}{p-k}\right) \\ &= \sum_{k=1}^{(p-1)/2} \frac{p}{k(p-k)} = p \sum_{k=1}^{(p-1)/2} \frac{1}{k(p-k)}. \end{aligned}$$

Since a p has been pulled out, we want to show the last sum is in $p\mathbf{Z}_p$. The terms in the sum are p-adic units, and reducing the terms modulo p yields

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k(p-k)} \equiv -\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \bmod p\mathbf{Z}_p.$$

The numbers $1^2, \ldots, ((p-1)/2)^2$ represent all the nonzero squares modulo p, so their reciprocals also represent the nonzero squares modulo p. Therefore

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \sum_{k=1}^{(p-1)/2} k^2 \bmod p\mathbf{Z}_p.$$

Using the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ with $n = (p-1)/2$,

$$\sum_{k=1}^{(p-1)/2} k^2 = \frac{p(p^2-1)}{24}.$$

Since $p > 3$, $(p, 24) = 1$, so this sum is in $p\mathbf{Z}_p$ and we're done. $\square$

Remark 6. Wolstenholme's paper writes $n!$ using the obsolete "corner" notation. See https://kconrad.math.uconn.edu/factorials/.

Theorem 5 says $\text{ord}_p(H_{p-1}) \geq 2$. If one calculates H_{p-1} for primes $3 < p < 10000$, always $\text{ord}_p(H_{p-1}) = 2$. But at $p = 16843$ and at $p = 2124679$, $\text{ord}_p(H_{p-1}) = 3$. No further examples are known where $\text{ord}_p(H_{p-1}) > 2$.

References

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