# OSTROWSKI'S THEOREM FOR $F(T)$

KEITH CONRAD

On $\mathbf{Q}$, Ostrowski's theorem says any nontrivial absolute value is a power of exactly one of the real and $p$-adic absolute values. For a field $F$, the rational function field $F(T)$ has a similar collection of nontrivial absolute values. For $0 < c < 1$, define

$$|r(T)|_{\infty,c} := c^{-\deg(r(T))}$$

when $r(T) \neq 0$, where $\deg(f/g) := \deg f - \deg g$ for nonzero $f$ and $g$ in $F[T]$, and $|0|_{\infty,c} = 0$. For each monic irreducible $\pi$ in $F[T]$ and $0 < c < 1$, define

$$|r(T)|_{\pi,c} := c^{\operatorname{ord}_\pi(r(T))}$$

when $r(T) \neq 0$, where $\operatorname{ord}_\pi(f/g) := \operatorname{ord}_\pi f - \operatorname{ord}_\pi g$ for nonzero $f$ and $g$ in $F[T]$, and $|0|_{\pi,c} = 0$.[1]

**Example 1.** Using $c = 1/2$, on $\mathbf{Q}(T)$ we have $|T^3 + T|_{\infty,c} = 8$, $|T^3 + T|_{T,c} = 1/2$, $|T^3 + T|_{T^2+1,c} = 1/2$, and $|T^3 + T|_{\pi,c} = 1$ when $\pi$ is not $T$ or $T^2 + 1$.

Why are $|\cdot|_{\infty,c}$ and $|\cdot|_{\pi,c}$ absolute values? The functions $|\cdot|_{\infty,c}$ and $|\cdot|_{\pi,c}$ are multiplicative since deg and $\operatorname{ord}_\pi$ on $F(T)^\times$ send products to sums (check that first on $F[T]-\{0\}$ and then extend to ratios). The function $|\cdot|_{\pi,c}$ satisfies the strong triangle inequality for the same reason $|\cdot|_p$ does on $\mathbf{Q}$: $\operatorname{ord}_\pi(f+g) \geq \min(\operatorname{ord}_\pi f, \operatorname{ord}_\pi g)$ when $f$, $g$, and $f+g$ are nonzero in $F[T]$, which extends to ratios in $F(T)^\times$ using common denominators, and raising $c$ to both sides of that inequality reverses the sense of the inequality since $0 < c < 1$ and that is also why $c^{\min(m,n)} = \max(c^m, c^n)$ for all integers $m$ and $n$. The function $|\cdot|_{\infty,c}$ on $F(T)$ satisfies the strong triangle inequality since $\deg(f + g) \leq \max(\deg f, \deg g)$ and raising $c^{-1} = 1/c$ to both sides retains the sense of the inequality with a maximum since $1/c > 1$. In particular, $|\cdot|_{\infty,c}$ on $F(T)$ is non-archimedean, in contrast to the usual absolute value on $\mathbf{Q}$.

**Remark 2.** Although the usual absolute value on $\mathbf{Q}$ and $|\cdot|_{\infty,c}$ on $F(T)$ are archimedean and non-archimedean, respectively, they are similar in one respect: they are both greater than 1 on most of $\mathbf{Z}$ and $F[T]$ while the other absolute values (trivial on $F$, in the second case) are less than or equal to 1 on all of $\mathbf{Z}$ and $F[T]$.[2]

Both $|\cdot|_{\infty,c}$ and $|\cdot|_{\pi,c}$ are trivial on $F$, but sometimes $F(T)$ has nontrivial absolute values that are nontrivial on $F$.

**Example 3.** Take $F = \mathbf{Q}$ and pick a transcendental real number $\alpha$. Embed $\mathbf{Q}(T)$ into $\mathbf{R}$ by substituting $\alpha$ for $T$, *i.e.*, $r(T) \mapsto r(\alpha)$ for any rational function $r(T)$. (To justify this, we should *first* evaluate only $\mathbf{Q}[T]$ at $\alpha$, getting a ring homomorphism $\mathbf{Q}[T] \to \mathbf{R}$. The

[1] These absolute values depend on the choice of $c$. Different $c$'s in $(0, 1)$ are powers of each other, so changing $c$ in an absolute value leads to an equivalent absolute value.

[2] While $\mathbf{Z}$ is a canonical subring of $\mathbf{Q}$, $F[T]$ is not a canonical subring of the field $F(T)$ since there's nothing canonical about $T$ in $F(T)$. For $U = 1/T$, $F(T) = F(U)$ and we could work with $F[U]$ in place of $F[T]$, which turns $|\cdot|_{\infty,c}$ into $|\cdot|_{U,c}$ (check!).

kernel is 0 since $\alpha$ is transcendental, so we can extend the homomorphism to all of $\mathbf{Q}(T)$. This would not work if $\alpha = \sqrt{2}$: where would $1/(T^2 - 2)$ go?)

Since $\mathbf{Q}(T)$ has been embedded in $\mathbf{R}$ using $\alpha$, setting $|r(T)| = |r(\alpha)|_\infty$ is an absolute value on $\mathbf{Q}(T)$ that is nontrivial on the subfield $\mathbf{Q}$ of $\mathbf{Q}(T)$.

**Example 4.** There is a similar example to the previous one using $p$-adic embeddings. Since $\mathbf{Q}$ is countable while $\mathbf{Q}_p$ is uncountable, a cardinality argument shows there are elements of $\mathbf{Q}_p$ that are transcendental (that is, not the root of a nonzero polynomial over $\mathbf{Q}$). Pick such a $p$-adic number $\alpha$ and set $|r(T)| = |r(\alpha)|_p$. As in the previous example, this is an absolute value on $\mathbf{Q}(T)$ that is nontrivial on $\mathbf{Q}$.

By focusing on absolute values on $F(T)$ that are *trivial* on $F$, we get an analogue of Ostrowski's theorem on $F(T)$ .

**Theorem 5.** *A nontrivial absolute value on $F(T)$ that is trivial on $F$ is $|\cdot|_{\infty,c}$ or $|\cdot|_{\pi,c}$ for some monic irreducible $\pi$ in $F[T]$ and some $c \in (0,1)$.*

*Proof.* We will adapt the proof of Ostrowski's theorem for $\mathbf{Q}$ in [1, p. 44] to $F(T)$.

Let $|\cdot|$ be a nontrivial absolute value on $F(T)$ that is trivial on $F$. It is non-archimedean since an absolute value $|\cdot|$ on a field $K$ is non-archimedean if and only if $|n| \leq 1$ for all $n$ in the prime subring of $K$ (the image of the unique ring homomorphism $\mathbf{Z} \to K$) [1, Theorem 2.2.2, p. 30], and when $K = F(T)$ its prime subring is in $F$.

For a nonzero polynomial $f(T) = c_0 + c_1T + \cdots + c_dT^d$ of degree $d$,

$$|f(T)| = |c_0 + c_1T + \cdots + c_dT^d|. \tag{1}$$

We will take two cases, depending on whether or not $|T| > 1$.

First assume $|T| > 1$. For all $i$, $|c_i|$ is 0 or 1, and $|c_d| = 1$ since $c_d \neq 0$. Therefore $|c_dT^d| = |T|^d$, while for $i < d$ we have $|c_iT^i| \leq |T|^i < |T|^d$. (Here we use $|T| > 1$.) Thus $c_dT^d$ has larger size than all the others terms in $f(T)$, which means by the non-archimedean property that $|f(T)| = |c_dT^d| = |T|^d = |T|^{\deg f}$. Let $c = 1/|T| < 1$, so $|f(T)| = c^{-\deg f}$.

For a nonzero rational function $r(T)$, write $r(T) = f(T)/g(T)$ with polynomials $f(T)$ and $g(T)$. Then $|r(T)| = |f(T)|/|g(T)| = c^{-\deg f + \deg g} = c^{-\deg(r(T))}$. Therefore $|\cdot| = |\cdot|_{\infty,c}$.

Now assume $|T| \leq 1$. Since $|\cdot|$ is non-archimedean, in the notation of (1) we have $|f(T)| \leq \max_i |c_iT^i| \leq 1$ for all $f(T) \in F[T]$. Since $|\cdot|$ is nontrivial and every rational function is a ratio of polynomials, some polynomial in $F[T]$ has to have absolute value not equal to 1 (and thus less than 1). It must be a nonconstant polynomial since $|\cdot|$ is trivial on $F$ (aha!). Let $\pi(T)$ be a nonconstant polynomial of least degree with $|\pi(T)| < 1$. The polynomial $\pi(T)$ is irreducible: if $\pi(T) = a(T)b(T)$ with $\deg a, \deg b < \deg \pi$ then $|\pi(T)| = |a(T)b(T)| = |a(T)||b(T)| = 1 \cdot 1 = 1$, which is a contradiction. Scaling by a nonzero element of $F$ doesn't change absolute values since the absolute value is trivial on $F$, so we may assume $\pi(T)$ is monic.

Set $c = |\pi(T)| < 1$. We will show $|\cdot| = |\cdot|_{\pi,c}$: for every $r(T) \in F(T)^\times$, $|r(T)| = c^{\operatorname{ord}_\pi(r(T))}$. This identity is multiplicative on both sides, so it is sufficient to prove it on $F[T] - \{0\}$.

For $f(T)$ in $F[T] - \{0\}$, write $f(T) = \pi(T)^k h(T)$, where $\pi(T) \nmid h(T)$. Then $|f(T)| = c^k|h(T)| = c^{\operatorname{ord}_\pi(f(T))}|h(T)|$, so it remains to check $|h(T)| = 1$. By the division algorithm in $F[T]$, $h(T) = \pi(T)q(T) + r(T)$ where $r(T)$ is nonzero and $\deg r < \deg \pi$. From the minimality condition defining $\pi(T)$, $|r(T)| = 1$. Therefore in the sum $\pi(T)q(T) + r(T)$, the first term has size $|\pi(T)||q(T)| \leq c < 1$ and the second term has size 1. Since the terms have different sizes, the non-archimedean property of $|\cdot|$ implies $|h(T)| = |\pi(T)q(T) + r(T)| = \max(|\pi(T)q(T)|, |r(T)|) = 1$. □

For finite $F$, we get the closest analogy to Ostrowski's theorem on $\mathbf{Q}$:

**Corollary 6.** *When $F$ is a finite field, every nontrivial absolute value on $F(T)$ is equivalent to $|\cdot|_{\infty,c}$ or to $|\cdot|_{\infty,c}$ for some monic irreducible $\pi(T)$ in $F[T]$ and $c \in (0,1)$.*

*Proof.* When $F$ is finite, every nonzero element of $F$ is a root of unity (if $q = |F|$ then $a^{q-1} = 1$ for $a \in F^\times$), so every absolute value on $F(T)$ is automatically trivial on $F$. □

There is a product formula for absolute values on $F(T)$ that are trivial on $F$, but it needs the "$c$" for each nontrivial absolute value on $F(T)$ chosen in a careful way. For instance, in Example 1 where $c = 1/2$ for the nontrivial absolute values on $\mathbf{Q}(T)$, the absolute value of $T^3+T$ is not 1 only for $|\cdot|_{\infty,c}$, $|\cdot|_{T,c}$, and $|\cdot|_{T^2+1,c}$, with $|T^3+T|_{\infty,c}|T^3+T|_{T,c}|T^3+T|_{T^2+1,c} = 8(1/2)(1/2) = 2 \neq 1$. To get a product formula, we need the $c$ for each $\pi$-adic absolute value to depend on the degree of $\pi$. Here is the fix.

**Theorem 7.** *Let $F$ be a field and pick $c \in (0,1)$. For $r(T) \in F(T)^\times$, set*

$$|r(T)|_\infty := c^{-\deg(r(T))} = |r(T)|_{\infty,c}, \quad |r(T)|_\pi := (c^{\deg \pi})^{\operatorname{ord}_\pi(r(T))} = |r(T)|_{\pi,c^{\deg\pi}},$$

*where $\pi(T)$ is a monic irreducible in $F[T]$. Then*

$$\prod_v |r(T)|_v = 1,$$

*where the product extends over all absolute values we just defined.*

*Proof.* For any nonzero $r(T)$, $|r(T)|_v = 1$ for all but finitely many of these absolute values (just $|\cdot|_\infty$ and $|\cdot|_\pi$ for any $\pi$ occurring in the numerator or denominator of $r(T)$). Since the product is a multiplicative function of $r(T)$ and the product formula is obvious on nonzero constants (each term in the product is 1), it suffices to check the product formula on monic polynomials, and in fact on monic irreducible polynomials. For a monic irreducible $\pi(T)$, the product has only two terms that are not 1, namely $|\pi(T)|_\infty$ and $|\pi(T)|_\pi$. Therefore

$$\prod_v |\pi(T)|_v = |\pi(T)|_\infty |\pi(T)|_\pi = c^{-\deg(\pi(T))}(c^{\deg\pi})^{\operatorname{ord}_\pi(\pi(T))} = c^{-\deg(\pi(T))}c^{\deg(\pi(T))} = 1. \quad \square$$

**Example 8.** On $\mathbf{Q}(T)$ let the infinite absolute value be $|\cdot|_{\infty,1/2}$ and let the $\pi$-adic absolute value be $|\cdot|_{\pi,1/2^{\deg\pi}}$. Then $\prod_v |T^3+T|_v = |T^3+T|_{\infty,1/2}|T^3+T|_{T,1/2}|T^3+T|_{T^2+1,1/4} = 8(1/2)(1/4) = 1$. Contrast this with what we saw in Example 1.

**Example 9.** When $F = \mathbf{F}_q$ is a finite field with size $q$, it is standard to take $c = 1/q$ in the definition of the infinite absolute value on $\mathbf{F}_q(T)$ and $c = 1/q^{\deg\pi}$ in the $\pi$-adic absolute value: for nonzero $r(T)$ in $\mathbf{F}_q(T)$, set

$$|r(T)|_\infty = \left(\frac{1}{q}\right)^{-\deg(r(T))} = q^{\deg(r(T))}, \quad |r(T)|_\pi = \left(\frac{1}{q^{\deg\pi}}\right)^{\operatorname{ord}_\pi(r(T))}.$$

There is a justification for these absolute values based on harmonic analysis over different completions of $\mathbf{F}_q(T)$, but we won't discuss that here.

The classification of the non-archimedean absolute values on $F(T)$ that are trivial on $F$ first treated the case where $|T| \leq 1$ and then the case where $|T| > 1$. The first case is the same as saying $|\cdot| \leq 1$ on $F[T]$, because $|\cdot|$ is assumed to be non-archimedean and trivial on $F$, and this proof proceeded in the same way as the non-archimedean case of Ostrowski's theorem for $\mathbf{Q}$. In particular, we use the division algorithm at one point. By a slight adjustment, the same argument can be carried over to the fraction field of any PID. Let $R$ be a PID with fraction field $K$. For any prime $\pi$ in $R$, we get a $\pi$-adic valuation on $R - \{0\}$: $\operatorname{ord}_\pi(x) = k$ is the largest nonnegative integer such that $\pi^k | x$ in $R$. That is, $x = \pi^k x'$ where $\pi$ doesn't divide $x'$. Then $\operatorname{ord}_\pi(x_1 x_2) = \operatorname{ord}_\pi(x_1) + \operatorname{ord}_\pi(x_2)$ for nonzero $x_1$ and $x_2$ in $R$, by unique factorization, so $\operatorname{ord}_\pi$ extends in a well-defined way from $R - \{0\}$ to $K^\times$ by $\operatorname{ord}_\pi(x/y) = \operatorname{ord}_\pi(x) - \operatorname{ord}_\pi(y)$. For any $c \in (0, 1)$, the function

$$|x| = \begin{cases} c^{\operatorname{ord}_\pi(x)}, & \text{if } x \in K^\times, \\ 0, & \text{if } x = 0 \end{cases}$$

is a non-archimedean absolute value on $K$, which we will call $\pi$-adic. It is $\leq 1$ on $R$, is nontrivial since $|\pi| < 1$, and $|\pi|$ is the largest value $< 1$ of $|\cdot|$ on $K$. We will now prove a converse result.

**Theorem 10.** *Let $R$ be a* PID *with fraction field $K$. Each nontrivial non-archimedean absolute value on $K$ that is $\leq 1$ on $R$ is a $\pi$-adic absolue value on $R$ for a prime $\pi$ that is unique up to multiplication by a unit.*

*Proof.* First we show $|\cdot|$ is trivial on the units of $R$. If $u \in R^\times$ then $|u| \leq 1$ and $|1/u| \leq 1$, so $|u| = 1$. Since $|\cdot|$ is nontrivial on $K$ and every element of $K^\times$ is a ratio of elements of $R - \{0\}$, $|\cdot|$ has a value $\neq 1$ at some nonzero element of $R$. Then, since every nonzero element of $R$ is a product of units and primes, $|\cdot|$ has a value $\neq 1$ at some unit or prime in $R$. Since $|\cdot| = 1$ on $R^\times$, we get $|\pi| < 1$ for some prime $\pi$.

Set $c = |\pi| < 1$. We will show $|x| = c^{\operatorname{ord}_\pi(x)}$ for all $x \in K^\times$. Both sides of this desired identity are multiplicative, so it is sufficient to check the identity on $R - \{0\}$.

For $x \in R - \{0\}$, write $x = \pi^k x'$, where $\pi \nmid x'$. Then $|x| = |\pi|^k |x'| = c^k |x'| = c^{\operatorname{ord}_\pi(x)} |x'|$, so it remains to check $|x'| = 1$. Since $\pi$ is a prime not dividing $x'$, $\pi$ and $x'$ are relatively prime. Then, since $R$ is a PID, we have $\pi u + x'v = 1$ for some $u$ and $v$ in $R$. The absolute value of 1 is 1, so $|\pi u + x'v| = 1$. Since $|\pi u| = |\pi||u| \leq |\pi| < 1$, we must have $|x'v| = 1$ by the non-archimedean property. Both $|x'|$ and $|v|$ are at most 1, so their product being 1 forces each to be 1. Thus $|x'| = 1$. This shows $|\cdot|$ is a $\pi$-adic absolute value.

Now we show $\pi$ is determined by $|\cdot|$ up to multiplication by a unit. If $|\cdot|$ is a $\pi'$-adic absolute value, then $|\pi'| < 1$. Any prime that is not a unit multiple of $\pi$ has $\operatorname{ord}_\pi$-value 0, so a $\pi$-adic absolute value of it is 1. Therefore $\pi'$ is a unit multiple of $\pi$. □

This theorem is *not* saying that every nontrivial non-archimedean absolute value on the fraction field of a PID $R$ is $\pi$-adic for some $\pi$: the condition that the absolute value be $\leq 1$ on $R$ might not be satisfied for all non-archimedean absolute values on $R$. For example, if $R = \mathbf{F}[T]$ for $\mathbf{F}$ a finite field then $|\cdot|_\infty$ is such an absolute value. But when $R = \mathbf{Z}$, any non-archimedean absolute value on $\mathbf{Q}$ is $\leq 1$ on $\mathbf{Z}$. Similarly, when $R = \mathbf{Z}[\sqrt{d}]$ for some non-square integer $d$, any non-archimedean absolute value on $\mathbf{Q}[\sqrt{d}]$ is $\leq 1$ on $\mathbf{Z}[\sqrt{d}]$ (because $|d| \leq 1$, so $|\sqrt{d}|^2 \leq 1$, so $|\sqrt{d}| \leq 1$, and then $|a + b\sqrt{d}| \leq \max(|a|, |b||\sqrt{d}|) \leq 1$ for all $a$ and $b$ in $\mathbf{Z}$). So if $\mathbf{Z}[\sqrt{d}]$ is a PID, which is the case for $d = -1, 2$, and $-2$, then each non-trivial non-archimedean absolute value on $\mathbf{Q}[\sqrt{d}]$ is $\pi$-adic for some prime $\pi$ in $\mathbf{Z}[\sqrt{d}]$.

## References

[1] F. Gouvea, "$p$-Adic Numbers: An Introduction," 2nd ed., Springer–Verlag, New York, 1997.