OSTROWSKI'S THEOREM FOR $F(T)$

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On $\mathbf{Q}$, Ostrowski's theorem says any nontrivial absolute value is a power of exactly one of the real and p-adic absolute values. For a field F, the rational function field $F(T)$ has a similar collection of absolute values: one for each monic irreducible in $F[T]$ and also an additional one associated to the degree function on $F[T]$ (where $-\deg(r(T))$ plays a role analogous to $\mathrm{ord}_\pi(r(T))$). Is every nontrivial absolute value on $F(T)$ a power of one of these basic examples? Not necessarily.

Example 1. Take $F = \mathbf{Q}$ and pick a transcendental real number α. Embed $\mathbf{Q}(T)$ into $\mathbf{R}$ by substituting α for T, *i.e.*, $r(T) \mapsto r(\alpha)$ for any rational function $r(T)$. (To justify this, we should *first* evaluate only polynomials at α, getting a ring homomorphism $\mathbf{Q}[T] \to \mathbf{R}$. Since α is transcendental, the kernel of this map is 0, and thus we can extend the map to the field $\mathbf{Q}(T)$. Notice this would not work if $\alpha = \sqrt{2}$: where would $1/(T^2-2)$ go?)

Since $\mathbf{Q}(T)$ has been embedded in $\mathbf{R}$, we get an absolute value on $\mathbf{Q}(T)$ by sending a rational function $r(T)$ to the real absolute value of the number $r(\alpha)$. That is, $|r(T)| = |r(\alpha)|_\infty$. We have defined an absolute value on $\mathbf{Q}(T)$ that is not one of those associated to irreducibles in $\mathbf{Q}[T]$ or to the (negative) degree function, since those absolute values are trivial on the subfield $\mathbf{Q}$ of $\mathbf{Q}(T)$ while this new absolute value we have put on $\mathbf{Q}(T)$ is nontrivial on $\mathbf{Q}$.

Example 2. One can build a similar example to the previous one using p-adic embeddings. Since the field $\mathbf{Q}$ is countable while $\mathbf{Q}_p$ is uncountable, a cardinality argument shows there do exist elements of $\mathbf{Q}_p$ that are transcendental (that is, not the root of a nonzero polynomial over $\mathbf{Q}$). Pick such a p-adic number α and set $|r(T)| = |r(\alpha)|_p$. Just as before, this is an absolute value on $\mathbf{Q}(T)$ and it is not trivial on $\mathbf{Q} \subset \mathbf{Q}(T)$, so it is not equivalent to one of the absolute values on $\mathbf{Q}(T)$ associated to irreducibles or to the (negative) degree.

Despite these examples, there is an analogue of Ostrowski's theorem on $F(T)$. We just need to introduce a condition on the absolute values to rule out the above examples. It is the assumption that the absolute value on $F(T)$ is trivial on F. This is not satisfied in the above examples and it is satisfied by the absolute values associated to irreducibles and the (negative) degree. With this condition made explicit, we can adapt the proof of Ostrowski's theorem in $\mathbf{Q}$, say from [1, p. 44], to the setting of $F(T)$.

Theorem 3. *Any nontrivial absolute value on $F(T)$ that is trivial on F is equivalent to $|\cdot|_\infty$ or to some $|\cdot|_\pi$ for monic irreducible π in $F[T]$.*

Proof. Let $|\cdot|$ be a nontrivial absolute value on $F(T)$ that is trivial on F. It is non-archimedean. For a nonzero polynomial $f(T) = c_0 + c_1T + \cdots + c_dT^d$ of degree d,

$$|f(T)| = |c_0 + c_1T + \cdots + c_dT^d|. \tag{1}$$

We will take two cases, depending on whether or not $|T| > 1$.

First assume $|T| > 1$. For all i, $|c_i|$ is 0 or 1, and $|c_d| = 1$ since $c_d \neq 0$. Therefore $|c_dT^d| = |T|^d$, while for $i < d$ we have $|c_iT^i| \leq |T|^i < |T|^d$. (Here we use $|T| > 1$.) Thus

c_dT^d has larger size than all the others terms in $f(T)$, which means by the non-archimedean property that $|f(T)| = |c_dT^d| = |T|^d = |T|^{\deg f}$. Let $c = 1/|T| < 1$, so $|f(T)| = c^{-\deg f}$.

For any nonzero rational function $r(T)$, write $r(T) = f(T)/g(T)$ with polynomials $f(T)$ and $g(T)$. Then $|r(T)| = |f(T)|/|g(T)| = c^{-\deg f+\deg g} = c^{-\deg(r(T))}$. Therefore $|\cdot| = |\cdot|_\infty$.

Now assume $|T| \leq 1$. Then by the non-archimedean property, in the notation of (1), $|f(T)| \leq \max_i |c_iT^i| \leq 1$, so all polynomials have absolute value less than or equal to 1. Since $|\cdot|$ is nontrivial and every rational function is a ratio of polynomials, some polynomial has to have absolute value not equal to 1 (and thus less than 1). It must be a nonconstant polynomial since the absolute value is trivial on constants (aha!). Let $\pi(T)$ be a nonconstant polynomial of least degree with $|\pi(T)| < 1$. The polynomial $\pi(T)$ is irreducible: if $\pi(T) = a(T)b(T)$ with $\deg a, \deg b < \deg \pi$ then $|\pi(T)| = |a(T)b(T)| = |a(T)||b(T)| = 1 \cdot 1 = 1$, a contradiction. Scaling by a nonzero element of F doesn't change absolute values since the absolute value is trivial on F, so we may assume $\pi(T)$ is monic.

Set $c = |\pi(T)| < 1$. We will show $|\cdot|$ is the π-adic absolute value: for any nonzero rational function $r(T)$, $|r(T)| = c^{\operatorname{ord}_\pi(r(T))}$. This identity is multiplicative on both sides, so it is sufficient to check the identity on polynomials.

For any $f(T)$ in $F[T]$, write $f(T) = \pi(T)^k h(T)$, where $h(T)$ is not divisible by $\pi(T)$. Then $|f(T)| = c^k|h(T)| = c^{\operatorname{ord}_\pi(f(T))}|h(T)|$, so it remains to check $|h(T)| = 1$. Using division in $F[T]$, we can write $f(T) = \pi(T)q(T)+r(T)$ where $r(T)$ is nonzero and $\deg r < \deg \pi$. From the minimality condition defining $\pi(T)$, $|r(T)| = 1$. Therefore in the sum $\pi(T)q(T) + r(T)$, the first term has size $|\pi(T)||q(T)| \leq c < 1$ and the second term has size 1. Since the terms have different sizes, from the non-archimedean property the size of the sum is the larger of the two sizes: $|h(T)| = |\pi(T)q(T) + r(T)| = 1$. $\square$

For finite F, we get the closest analogy to Ostrowski's theorem on $\mathbf{Q}$:

Corollary 4. *When F is a finite field, every nontrivial absolute value on $F(T)$ is equivalent to $|\cdot|_\infty$ or to $|\cdot|_\pi$ for some monic irreducible $\pi(T)$ in $F[T]$.*

Proof. When F is finite, every nonzero element of F is a root of unity (if $q = \#F$ then $a^{q-1} = 1$ for $a \in F^\times$), so any absolute value on $F(T)$ is automatically trivial on F. $\square$

Remark 5. The absolute values denoted $|\cdot|_\infty$ on $\mathbf{Q}$ and $F(T)$ are similar in one respect: they are greater than 1 on $\mathbf{Z}$ and $F[T]$ while the other absolute values (trivial on F, in the second case) are all less than or equal to 1 on $\mathbf{Z}$ and $F[T]$. However, there is still a big difference: the absolute value $|\cdot|_\infty$ on $\mathbf{Q}$ is not non-archimedean, while $|\cdot|_\infty$ on $F(T)$ is non-archimedean. There's also another distinction: $\mathbf{Z}$ is a canonical subring of $\mathbf{Q}$, but $F[T]$ is not in anyway canonical from the viewpoint of the field $F(T)$: what's so special about T? After all, taking $U = 1/T$ we have $F(T) = F(U)$ so we could work with $F[U]$ in place of $F[T]$. Now the old $|\cdot|_\infty$ becomes $|\cdot|_U$ (check!).

To get a product formula on $F(T)$, we need to choose constants for the absolute values in a compatible way. For instance, using $|r| = c^{-\deg r}$ and $|r|_\pi = c^{\operatorname{ord}_\pi(r)}$ won't yield a product formula for all $r \neq 0$. We need to tweak the constant for each $|\cdot|_\pi$ in a way that reflects the degree of π. Here is the fix:

Theorem 6. *Let F be a field and pick $c \in (0, 1)$. For $r(T) \in F(T)$, set*

$$|r(T)|_\infty = c^{-\deg(r(T))}, \quad |r(T)|_\pi = (c^{\deg \pi})^{\operatorname{ord}_\pi(r(T))},$$

where $\pi(T)$ is a monic irreducible in $F[T]$. Then for any $r(T) \neq 0$ in $F(T)$,

$$\prod_v |r(T)|_v = 1,$$

where the product extends over all absolute values we just defined.

Proof. For any nonzero $r(T)$, $|r(T)|_v = 1$ for all but finitely many of these absolute values (just $|\cdot|_\infty$ and $|\cdot|_\pi$ for any π occurring in the numerator or denominator of $r(T)$). Since the product is a multiplicative function of $r(T)$ and the product formula is obvious on nonzero constants (each term in the product is 1), it suffices to check the product formula on monic polynomials, and in fact on monic irreducible polynomials. For a monic irreducible $\pi(T)$, the product has only two terms that are not 1, namely $|\pi(T)|_\infty$ and $|\pi(T)|_\pi$. Therefore

$$\prod_v |\pi(T)|_v = |\pi(T)|_\infty |\pi(T)|_\pi = c^{-\deg(\pi(T))}(c^{\deg \pi})^{\mathrm{ord}_\pi(\pi(T))} = c^{-\deg(\pi(T))}c^{\deg(\pi(T))} = 1. \quad \square$$

When $F = \mathbf{F}_q$ is a finite field with size q, it is standard to take $c = 1/q$ in the definition of the absolute values on $\mathbf{F}_q(T)$:

$$|r(T)|_\infty = \left(\frac{1}{q}\right)^{-\deg(r(T))}, \quad |r(T)| = \left(\frac{1}{q^{\deg \pi}}\right)^{\mathrm{ord}_\pi(r(T))}.$$

This choice of c could be considered as simply a convention, but there is genuine reason for this choice of c in the setting of harmonic analysis over the completions of $\mathbf{F}_q(T)$.

The classification of the non-archimedean absolute values on $F(T)$ which are trivial on F first treated the case where $|T| \leq 1$ and then the case where $|T| > 1$. The first case is the same as saying $|\cdot| \leq 1$ on $F[T]$, because $|\cdot|$ is assumed to be non-archimedean and trivial on F, and this proof proceeded in the same way as the non-archimedean case of Ostrowski's theorem for $\mathbf{Q}$. In particular, we use the division theorem at one point. By a slight adjustment, the same argument can be carried over to the fraction field of any PID. Let R be a PID with fraction field K. For any prime π in R, we get a π-adic valuation on $R - \{0\}$: $\mathrm{ord}_\pi(x) = k$ is the largest nonnegative integer such that $\pi^k | x$ in R. That is, $x = \pi^k x'$ where π doesn't divide x'. Then $\mathrm{ord}_\pi(x_1x_2) = \mathrm{ord}_\pi(x_1) + \mathrm{ord}_\pi(x_2)$ for nonzero x_1 and x_2 in R, by unique factorization, so ord_π extends in a well-defined way from $R - \{0\}$ to $K^\times$ by $\mathrm{ord}_\pi(x/y) = \mathrm{ord}_\pi(x) - \mathrm{ord}_\pi(y)$. For any $c \in (0,1)$, the function

$$|x| = \begin{cases} c^{\mathrm{ord}_\pi(x)}, & \text{if } x \in K^\times, \\ 0, & \text{if } x = 0 \end{cases}$$

is a non-archimedean absolute value on K, which we will call a π-adic absolute value on K. By construction, a π-adic absolute value is ≤ 1 on R and is nontrivial since $|\pi| < 1$. Moreover, $|\pi|$ is the largest value < 1 of $|\cdot|$ on K. If π' is a unit multiple of π, then $\mathrm{ord}_{\pi'} = \mathrm{ord}_\pi$, so the notions of π-adic and π'-adic absolute values are the same. We will now prove a converse result for such a construction, which can be viewed as essentially a generalized Ostrowski theorem.

Theorem 7. *Let R be a* PID *with fraction field K. Any nontrivial non-archimedean absolute value on K that is ≤ 1 on R is a π-adic absolue value on R for a prime π that is unique up to multiplication by a unit.*

Proof. First we show $|\cdot|$ is trivial on the units of R. If $u \in R^\times$ then $|u| \leq 1$ and $|1/u| \leq 1$, so $|u| = 1$. Since $|\cdot|$ is nontrivial on K and every element of $K^\times$ is a ratio of elements of

$R - \{0\}$, $|\cdot|$ has a value $\neq 1$ at some nonzero element of R. Then, since every nonzero element of R is a product of units and primes, $|\cdot|$ has a value $\neq 1$ at some unit or prime in R. Since $|\cdot| = 1$ on $R^\times$, we get $|\pi| < 1$ for some prime π.

Set $c = |\pi| < 1$. We will show $|x| = c^{\mathrm{ord}_\pi(x)}$ for all $x \in K^\times$. Both sides of this desired identity are multiplicative, so it is sufficient to check the identity on $R - \{0\}$.

For any x in R, write $x = \pi^k x'$, where x' is not divisible by π. Then $|x| = |\pi|^k|x'| = c^k|x'| = c^{\mathrm{ord}_\pi(x)}|x'|$, so it remains to check $|x'| = 1$. Since π is a prime not dividing x', π and x' are relatively prime. Then, since R is a PID, we have $\pi u + x'v = 1$ for some u and v in R. The absolute value of 1 is 1, so $|\pi u + x'v| = 1$. Since $|\pi u| = |\pi||u| \leq |\pi| < 1$, we must have $|x'v| = 1$ by the non-archimedean property. Both $|x'|$ and $|v|$ are at most 1, so their product being 1 forces each to be 1, so $|x'| = 1$. This shows $|\cdot|$ is a π-adic absolute value.

Now we show π is determined by $|\cdot|$ up to multiplication by a unit. If $|\cdot|$ is a π'-adic absolute value, then $|\pi'| < 1$. Any prime that is not a unit multiple of π has ord_π-value 0, so a π-adic absolute value of it is 1. Therefore π' has to be a unit multiple of π. $\square$

This theorem is *not* saying that every nontrivial non-archimedean absolute value on the fraction field of a PID R is π-adic for some π: the condition that the absolute value be ≤ 1 on R might not be satisfied for all non-archimedean absolute values on R. For example, if $R = \mathbf{F}[T]$ for $\mathbf{F}$ a finite field then $|\cdot|_\infty$ is such an absolute value. But when $R = \mathbf{Z}$, any non-archimedean absolute value on $\mathbf{Q}$ is ≤ 1 on $\mathbf{Z}$. Similarly, when $R = \mathbf{Z}[\sqrt{d}]$ for some non-square integer d, any non-archimedean absolute value on $\mathbf{Q}[\sqrt{d}]$ is ≤ 1 on $\mathbf{Z}[\sqrt{d}]$ (because $|d| \leq 1$, so $|\sqrt{d}|^2 \leq 1$, so $|\sqrt{d}| \leq 1$, and then $|a + b\sqrt{d}| \leq \max(|a|, |b||\sqrt{d}|) \leq 1$ for all a and b in $\mathbf{Z}$). So if $\mathbf{Z}[\sqrt{d}]$ is a PID, which is the case for $d = -1, 2$, and -2, then we know that all non-trivial non-archimedean absolute values on $\mathbf{Q}[\sqrt{d}]$ are π-adic for some prime π in $\mathbf{Z}[\sqrt{d}]$.

References

[1] F. Gouvea, "p-Adic Numbers: An Introduction," 2nd ed., Springer–Verlag, New York, 1997.