A NON-FREE RELATIVE INTEGRAL EXTENSION

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1. Introduction

The ring of integers of any algebraic number field is free as a $\mathbf{Z}$-module. More precisely, if $[K:\mathbf{Q}] = n$, then there are $\omega_1, \dots, \omega_n$ such that

$$\mathcal{O}_K = \mathbf{Z}\omega_1 \oplus \cdots \oplus \mathbf{Z}\omega_n.$$

We call $\omega_1, \dots, \omega_n$ an *integral basis* for $K/\mathbf{Q}$.

The existence of an integral basis is proved by showing $\mathcal{O}_K$ both *contains* a free rank-n $\mathbf{Z}$-module and (via discriminants) is *contained in* a free rank-n $\mathbf{Z}$-module as well. Therefore $\mathcal{O}_K$ is also a free rank-n $\mathbf{Z}$-module by the theory of modules over a PID. (Any module over a PID which is stuck between two finite free modules of the same rank is also finite free of that rank.)

Let's consider any finite extension of number fields E/F, where F may not be $\mathbf{Q}$. How does $\mathcal{O}_E$ look as an $\mathcal{O}_F$-module? Since $\mathcal{O}_E$ is finitely generated as a $\mathbf{Z}$-module so it is certainly also finitely generated as an $\mathcal{O}_F$-module:

$$\mathcal{O}_E = \mathcal{O}_F x_1 + \cdots + \mathcal{O}_F x_r \tag{1.1}$$

for some $x_1, \dots, x_r \in \mathcal{O}_E$. Can we make this a direct sum? If $\mathcal{O}_F$ is a PID, then this is possible, by the same proof as over $\mathbf{Z}$. (Use traces from E to F instead of from K to $\mathbf{Q}$.) However, if $\mathcal{O}_F$ is not a PID, then $\mathcal{O}_E$ need not be a free $\mathcal{O}_F$-module. We will give a family of examples in Theorem 2.2 below. It is inspired by the particular example in [1]. The argument used in [1] involves unique factorization of ideals into prime ideals, but we avoid this. (We will use ideal factorization to study the situation further, once examples are established.)

2. The Example

Lemma 2.1. *Let E/F be an extension of number fields. If $\mathcal{O}_E$ is a free $\mathcal{O}_F$-module, then $\mathcal{O}_E$ has rank $[E:F]$ over $\mathcal{O}_F$.*

Proof. If $\mathcal{O}_E$ is a free $\mathcal{O}_F$-module then we can choose the x_i's in (1.1) to make that sum a direct sum. Then r, in (1.1), is the rank of $\mathcal{O}_E$ as an $\mathcal{O}_F$-module. Since $\mathcal{O}_F$ has rank $[F:\mathbf{Q}]$ as a $\mathbf{Z}$-module, $\mathcal{O}_E$ has rank $r[F:\mathbf{Q}]$ as a $\mathbf{Z}$-module. We already know $\mathcal{O}_E$ has rank $[E:\mathbf{Q}]$ as a $\mathbf{Z}$-module, so $r[F:\mathbf{Q}] = [E:\mathbf{Q}]$. Thus $r = [E:F]$. $\square$

Here is the example.

Theorem 2.2. *Let $d \geq 2$ be a squarefree positive integer and q be a prime not dividing d. Assume $q \equiv 3 \bmod 4$. Let $F = \mathbf{Q}(\sqrt{-dq})$ and $E = F(\sqrt{-q}) = \mathbf{Q}(\sqrt{-dq}, \sqrt{-q})$. Then $\mathcal{O}_E$ is not free as an $\mathcal{O}_F$-module.*

Proof. If $\mathcal{O}_E$ is a free $\mathcal{O}_F$-module, then its rank is $[E:F]=2$ by Lemma 2.1.

We are less familiar with $E/F = F(\sqrt{-q})/F$ than we are with $\mathbf{Q}(\sqrt{-q})/\mathbf{Q}$. In particular, since $q \equiv 3 \bmod 4$ we have $-q \equiv 1 \bmod 4$, so $\mathbf{Q}(\sqrt{-q})/\mathbf{Q}$ has integral basis $\{1,(1+\sqrt{-q})/2\}$. Could this also be an integral basis for E/F?

Step 1: If $\mathcal{O}_E$ is a free $\mathcal{O}_F$-module, then $\{1,(1+\sqrt{-q})/2\}$ is an $\mathcal{O}_F$-basis of $\mathcal{O}_E$.

Suppose $\mathcal{O}_E = \mathcal{O}_F e_1 \oplus \mathcal{O}_F e_2$ for some e_1 and e_2 in $\mathcal{O}_E$. Both $\{1,(1+\sqrt{-q})/2\}$ and $\{e_1,e_2\}$ are F-bases of E. Since $\{e_1,e_2\}$ is a basis of $\mathcal{O}_E$ over $\mathcal{O}_F$ we have

$$\begin{aligned} 1 &= \alpha_1 e_1 + \alpha_2 e_2, \\ \frac{1+\sqrt{-q}}{2} &= \beta_1 e_1 + \beta_2 e_2, \end{aligned}$$

where the α_i's and β_i's are in $\mathcal{O}_F$. We will show the matrix $\left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\right)$ has determinant in $\mathcal{O}_F^\times$, so $\{1,(1+\sqrt{-q})/2\}$ is an $\mathcal{O}_F$-basis of $\mathcal{O}_E$ because $\{e_1,e_2\}$ is one and the matrix passing the latter pair to the former is in $\mathrm{GL}_2(\mathcal{O}_F)$.

The extension E/F is Galois, with non-trivial automorphism σ determined by $\sigma(\sqrt{-q}) = -\sqrt{-q}$. Since σ fixes elements of F, applying σ to the above equations yields

$$\begin{aligned} 1 &= \alpha_1 \sigma(e_1) + \alpha_2 \sigma(e_2), \\ \frac{1-\sqrt{-q}}{2} &= \beta_1 \sigma(e_1) + \beta_2 \sigma(e_2). \end{aligned}$$

We can collect all four equations into a matrix equation

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} e_1 & \sigma(e_1) \\ e_2 & \sigma(e_2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{-q}}{2} & \frac{1-\sqrt{-q}}{2} \end{pmatrix}.$$

Take the determinant of both sides:

$$(\alpha_1\beta_2 - \alpha_2\beta_1)(e_1\sigma(e_2) - \sigma(e_1)e_2) = -\sqrt{-q}. \tag{2.1}$$

The first term on the left side is the determinant we want to show is in $\mathcal{O}_F^\times$.

Both differences on the left side of (2.1) are algebraic integers in E. Are they in $\mathcal{O}_F$? The first difference is in $\mathcal{O}_F$ because every term in it is in $\mathcal{O}_F$. The second difference is non-zero and is negated after applying σ, so the second term is not in $\mathcal{O}_F$.

Now square both sides:

$$(\alpha_1\beta_2 - \alpha_2\beta_1)^2(e_1\sigma(e_2) - \sigma(e_1)e_2)^2 = -q. \tag{2.2}$$

The second squared term is now in $\mathcal{O}_F$, since it is σ-invariant. Thus (2.2) says $x^2 y = -q$ with $x = \alpha_1\beta_2 - \alpha_2\beta_1$ and $y = (e_1\sigma(e_2) - \sigma(e_1)e_2)^2$. We want to show $x \in \mathcal{O}_F^\times$.

Taking norms on (2.2) from F down to $\mathbf{Q}$, we get

$$\mathrm{N}_{F/\mathbf{Q}}(x)^2 \mathrm{N}_{F/\mathbf{Q}}(y) = q^2, \tag{2.3}$$

where x and y have norms in $\mathbf{Z}$ since x and y are algebraic integers. As F is imaginary quadratic, the norm from F to $\mathbf{Q}$ takes only non-negative values, and q is prime, so from (2.3) we see $\mathrm{N}_{F/\mathbf{Q}}(x)$ is either 1 or q. If $\mathrm{N}_{F/\mathbf{Q}}(x) = 1$ then $x = \alpha_1\beta_2 - \alpha_2\beta_1$ is a unit in $\mathcal{O}_F$ and we're done with Step 1. Thus, assume instead that $\mathrm{N}_{F/\mathbf{Q}}(x) = q$. We will get a contradiction.

Since $F = \mathbf{Q}(\sqrt{-dq})$, either $\mathcal{O}_F = \mathbf{Z}[\sqrt{-dq}]$ (if $-dq \not\equiv 1 \bmod 4$) or $\mathcal{O}_F = \mathbf{Z}[(1+\sqrt{-dq})/2]$ (if $-dq \equiv 1 \bmod 4$). In the first case the norm from $\mathcal{O}_F$ to $\mathbf{Z}$ has the form $a^2 + dqb^2$ for $a, b \in \mathbf{Z}$, and this never takes the value q. In the second case the norm from $\mathcal{O}_F$ to $\mathbf{Z}$ has the form $(a + \frac{1}{2}b)^2 + \left(\frac{b}{2}\right)^2 dq$. For $|b| \geq 2$ this norm value is at least $dq > q$. For $b = 0$

the norm value is a^2, which is never q. For $|b| = 1$ this norm value is at least $\frac{1+dq}{4}$, which exceeds q unless $d = 2$ or $d = 3$. But then $-dq \equiv d \not\equiv 1 \bmod 4$ so this is not the second case anyway. This concludes Step 1.

Step 2: $\{1, (1+\sqrt{-q})/2\}$ is *not* an $\mathcal{O}_F$-basis of $\mathcal{O}_E$.

Assume it is an $\mathcal{O}_F$-basis of $\mathcal{O}_E$. Since $\sqrt{d} := \sqrt{-dq}/\sqrt{-q} \in \mathcal{O}_E$, we must be able to write

$$\frac{\sqrt{-dq}}{\sqrt{-q}} = \alpha + \beta\left(\frac{1+\sqrt{-q}}{2}\right) \tag{2.4}$$

for some α and β in $\mathcal{O}_F$. Applying σ, the non-trivial automorphism of E fixing F,

$$-\frac{\sqrt{-dq}}{\sqrt{-q}} = \alpha + \beta\left(\frac{1-\sqrt{-q}}{2}\right). \tag{2.5}$$

Subtract (2.5) from (2.4):

$$\frac{2\sqrt{-dq}}{\sqrt{-q}} = \beta\sqrt{-q}.$$

Clearing the denominator and squaring,

$$-4dq = \beta^2 q^2.$$

Therefore $\beta^2 = -4d/q$, but $-4d/q$ is not an algebraic integer. $\square$

Example 2.3. Let $d = 2$ and $q = 3$. Theorem 2.2 says $\mathbf{Q}(\sqrt{-6}, \sqrt{-3})/\mathbf{Q}(\sqrt{-6})$ does not have an integral basis. (That is, the integers of $\mathbf{Q}(\sqrt{-6}, \sqrt{-3})$ do not have a basis over the integers of $\mathbf{Q}(\sqrt{-6})$.)

The quadratic field $\mathbf{Q}(\sqrt{2})$ is inside $\mathbf{Q}(\sqrt{-6}, \sqrt{-3}) = \mathbf{Q}(\sqrt{2}, \sqrt{-3})$ and its ring of integers $\mathbf{Z}[\sqrt{2}]$ is a PID, so the ring of integers of $\mathbf{Q}(\sqrt{-6}, \sqrt{-3})$ must have a basis over $\mathbf{Z}[\sqrt{2}]$. An example is $\{1, (1+\sqrt{-3})/2\}$ because if K and L are number fields with relatively prime discriminants, then $\mathcal{O}_{KL} = \mathcal{O}_K\mathcal{O}_L$: use the fields $K = \mathbf{Q}(\sqrt{2})$ and $L = \mathbf{Q}(\sqrt{-3})$, which have discriminants 8 and -3. Therefore $\mathcal{O}_{\mathbf{Q}(\sqrt{2},\sqrt{-3})} = \mathbf{Z}[\sqrt{2}, (1+\sqrt{3})/2] = \mathbf{Z}[\sqrt{2}][(1+\sqrt{-3})/2]$.

Example 2.4. Let $d = 5$ and $q = 3$. By Theorem 2.2, $\mathbf{Q}(\sqrt{-15}, \sqrt{-3})/\mathbf{Q}(\sqrt{-15})$ does not have an integral basis.

The quadratic subfield $\mathbf{Q}(\sqrt{5})$ of $\mathbf{Q}(\sqrt{-15}, \sqrt{-3}) = \mathbf{Q}(\sqrt{5}, \sqrt{-3})$ has ring of integers $\mathbf{Z}[(1+\sqrt{5})/2]$ and $\mathbf{Q}(\sqrt{5}, \sqrt{-3})/\mathbf{Q}(\sqrt{5})$ has integral basis $\{1, (1+\sqrt{-3})/2\}$ by the same reasoning as that used at the end of Example 2.3.

Example 2.5. The example in [1] uses $d = 2$ and $q = 7$: $\mathbf{Q}(\sqrt{-14}, \sqrt{-7})$ has no integral basis over $\mathbf{Q}(\sqrt{-14})$. In contrast to that, $\mathbf{Q}(\sqrt{-14}, \sqrt{-7}) = \mathbf{Q}(\sqrt{2}, \sqrt{-7})$ has integral basis $\{1, (1+\sqrt{-7})/2\}$ over $\mathbf{Q}(\sqrt{2})$.

Example 2.6. Let $d = 10$ and $q = 23$. The extension $\mathbf{Q}(\sqrt{-230}, \sqrt{-23})/\mathbf{Q}(\sqrt{-230})$ does not have an integral basis.

The quadratic subfields of $\mathbf{Q}(\sqrt{-230}, \sqrt{-23})$ are $\mathbf{Q}(\sqrt{-230})$, $\mathbf{Q}(\sqrt{-23})$, and $\mathbf{Q}(\sqrt{10})$. The second and third quadratic fields each have a ring of integers that is not a PID since their class numbers are greater than 1 (they are 3 and 2, respectively), so we have no reason to expect *a priori* that the ring of integers $\mathbf{Q}(\sqrt{-230}, \sqrt{-23}) = \mathbf{Q}(\sqrt{10}, \sqrt{-23})$ has an integral basis over the ring of integers of $\mathbf{Q}(\sqrt{-23})$ or $\mathbf{Q}(\sqrt{10})$. However, since $\mathbf{Q}(\sqrt{10})$ and $\mathbf{Q}(\sqrt{-23})$ have relatively prime discriminants (40 and-23, respectively), the ring of integers $\mathbf{Q}(\sqrt{10}, \sqrt{-23})$ is $\mathbf{Z}[\sqrt{10}, (1+\sqrt{-23})/2]$ and therefore this ring has a basis over $\mathbf{Z}[\sqrt{10}]$ and $\mathbf{Z}[(1+\sqrt{-23})/2]$.

Corollary 2.7. *Let $d \geq 2$ be a squarefree positive integer and q be a prime not dividing d with $q \equiv 3 \bmod 4$. The ring of integers of $\mathbf{Q}(\sqrt{-dq})$ is not a* PID.

Proof. We give two proofs. First, Theorem 2.2 constructs a finitely generated torsion-free module over the integer ring of $\mathbf{Q}(\sqrt{-dq})$ which is not a free module. Therefore the integers of $\mathbf{Q}(\sqrt{-dq})$ is not a PID. (Any finitely generated torsion-free module over a PID is free.) Second, we will explicitly write down a non-principal ideal. Since d and q are relatively prime, we obtain the equality of ideals

$$(q, \sqrt{-dq})^2 = (q) \tag{2.6}$$

in the integer ring of $\mathbf{Q}(\sqrt{-dq})$. The integer ring has unit group ± 1, and there is no solution to $\pm\alpha^2 = q$ in $\mathbf{Q}(\sqrt{-dq})$, so the ideal $(q, \sqrt{-pq})$ is not principal. □

3. Non-free Module Structure

In Theorem 2.2, $\mathcal{O}_E$ is not a free module over $\mathcal{O}_F$. What kind of description can we give for $\mathcal{O}_E$ as an $\mathcal{O}_F$-module?

Theorem 3.1. *Let $d \geq 2$ be a squarefree positive integer and q be a prime not dividing d with $q \equiv 3 \bmod 4$. Let $F = \mathbf{Q}(\sqrt{-dq})$ and $E = F(\sqrt{-q}) = \mathbf{Q}(\sqrt{-dq}, \sqrt{-q})$. The*

$$\mathcal{O}_E = \mathcal{O}_F e_1 \oplus \mathfrak{q} e_2, \tag{3.1}$$

where $e_1 = (1+\sqrt{-q})/2$, $e_2 = 1/\sqrt{-q}$, and $\mathfrak{q} = (q, \sqrt{-dq}) = q\mathcal{O}_F + \sqrt{-dq}\mathcal{O}_F$.

Thus, as an $\mathcal{O}_F$-module, $\mathcal{O}_E$ is isomorphic to a direct sum of two $\mathcal{O}_F$-modules, but one of the $\mathcal{O}_F$-modules is not free.

Proof. We will work with the E/F-basis $\{(1+\sqrt{-q})/2, \sqrt{-q}\}$, and see what constraints on coefficients make elements integral. At the end of the proof, we will see how it is natural to replace $\sqrt{-q}$ with $1/\sqrt{-q}$ in the basis.

Why do we pick this basis, rather than, say, $\{1, (1+\sqrt{-q})/2\}$? An advantage is that the trace of our basis elements, from E down to F, are 1 and 0 rather than 2 and 1. This will make it easier to figure out coefficient constraints on algebraic integers.

(Incidentally, the source of the contradiction at the very end of the proof of Theorem 2.2 was a denominator of q. Therefore, it is no surprise that a "good" spanning set for $\mathcal{O}_E$ as an $\mathcal{O}_F$-module is going to involve some q-related denominators.)

Write

$$\alpha = x\frac{1+\sqrt{-q}}{2} + y\sqrt{-q},$$

where $x, y \in F$. Assume $\alpha \in \mathcal{O}_E$. Then its trace and norm down to F must lie in $\mathcal{O}_F$: $\alpha + \overline{\alpha} \in F$ and $\alpha\overline{\alpha} \in \mathcal{O}_F$. In terms of coefficients, this says

$$x \in \mathcal{O}_F, \quad \frac{1+q}{4}x^2 + qxy + qy^2 \in \mathcal{O}_F,$$

which is equivalent to

$$x \in \mathcal{O}_F, \quad qxy + qy^2 \in \mathcal{O}_F$$

since $q \equiv 3 \bmod 4$. Let $z = qxy + qy^2$. Since qx and z are in $\mathcal{O}_F$, the equation

$$qy^2 + qxy - z = 0$$

nearly exhibits y as an algebraic integer. There is a coefficient of q out front, which we can collect with y by multiplying through by q:

$$(qy)^2 + qx(qy) - qz = 0.$$

Thus qy is integral over $\mathcal{O}_F$. Since $qy \in F$, we get $qy \in \mathcal{O}_F$. Therefore

$$qy^2 = z - qxy = z - x(qy) \in \mathcal{O}_F.$$

We view this as a product of ideals: $(q)(y)^2 \subset (1)$. By (2.6), $(q) = \mathfrak{q}^2$. Therefore $\mathfrak{q}^2(y)^2$ is an integral ideal, so $\mathfrak{q}(y)$ is an integral ideal, which means

$$(y) \subset \mathfrak{q}^{-1} = \mathfrak{q}\mathfrak{q}^{-2} = \frac{1}{q}\mathfrak{q}.$$

We have shown

$$x\frac{1+\sqrt{-q}}{2} + y\sqrt{-q} \in \mathcal{O}_E \Longrightarrow x \in \mathcal{O}_F, \quad y \in \mathfrak{q}^{-1} = \frac{1}{q}\mathfrak{q}.$$

Now we check the converse. Since $(1+\sqrt{-q})/2$ is an algebraic integer, so is $x(1+\sqrt{-q})/2$ when $x \in \mathcal{O}_F$. To see that $y\sqrt{-q}$ is an algebraic integer when $y \in \mathfrak{q}^{-1}$, it is simpler to look at its square, which is $-y^2q$. Since $y^2 \in \mathfrak{q}^{-2} = (1/q)\mathcal{O}_F$, we have $y^2q \in \mathcal{O}_F$. At last, we can write

$$\mathcal{O}_E = \mathcal{O}_F\left(\frac{1+\sqrt{-q}}{2}\right) \oplus \frac{1}{q}\mathfrak{q}\sqrt{-q} = \mathcal{O}_F\left(\frac{1+\sqrt{-q}}{2}\right) \oplus \mathfrak{q}\frac{1}{\sqrt{-q}}.$$

□

Remark 3.2. One has to be careful when using parentheses to denote "ideal generated by" if there are several rings floating around. For instance, in the notation of Theorem 3.1, we have $(q) = \mathfrak{q}^2$. In $\mathcal{O}_E$, where there is a square root of $-q$, we have $(q) = (\sqrt{-q})^2$. Therefore $\mathfrak{q} = (\sqrt{-q})$, and then $\mathfrak{q}/\sqrt{-q} = (1) = \mathcal{O}_E$, but (3.1) shows $\mathfrak{q}/\sqrt{-q} = \mathfrak{q}e_2$ is only one piece of $\mathcal{O}_E$. What went wrong?

To compare ideals, such as $\mathfrak{q}$ and $(\sqrt{-q}) = \sqrt{-q}\mathcal{O}_E$, they must be ideals in the same ring. The ideal $\mathfrak{q} = (q, \sqrt{-dq}) = q\mathcal{O}_F + \sqrt{-dq}\mathcal{O}_F$ was defined as an ideal in $\mathcal{O}_F$. To compare $\mathfrak{q}$ to $\sqrt{-q}\mathcal{O}_E$, we must extend $\mathfrak{q}$ to $\mathcal{O}_E$:

$$q\mathcal{O}_F = \mathfrak{q}^2 \Longrightarrow q\mathcal{O}_E = (\mathfrak{q}\mathcal{O}_E)^2.$$

Therefore, it is true that $\mathfrak{q}\mathcal{O}_E = \sqrt{-q}\mathcal{O}_E$, since both ideals *of* $\mathcal{O}_E$ square to $q\mathcal{O}_E$. Dividing now by $\sqrt{-q}$, we get

$$\frac{1}{\sqrt{-q}}\mathfrak{q}\mathcal{O}_E = \mathcal{O}_E. \tag{3.2}$$

There is no contradiction between (3.1) and (3.2), since we have the extended ideal $\mathfrak{q}\mathcal{O}_E$ on the left side of (3.2).

To see that (3.2) is true computationally, we will exhibit $\sqrt{-q}$ as an element of $\mathfrak{q}\mathcal{O}_E$:

$$\mathfrak{q}\mathcal{O}_E = (q\mathcal{O}_F + \sqrt{-dq}\mathcal{O}_F)\mathcal{O}_E = q\mathcal{O}_E + \sqrt{-dq}\mathcal{O}_E = \sqrt{-q}(\sqrt{-q}\mathcal{O}_E + \sqrt{d}\mathcal{O}_E), \tag{3.3}$$

where $\sqrt{d} := \sqrt{-dq}/\sqrt{-q}$ (some square root of d). Since d and q are relatively prime integers, we can write $1 = da - qb$ for some $a, b \in \mathbf{Z}$. Then

$$1 = \sqrt{-q}\sqrt{-q}b + \sqrt{d}\sqrt{d}a \in \sqrt{-q}\mathcal{O}_E + \sqrt{d}\mathcal{O}_E,$$

so the $\mathcal{O}_E$-ideal $(\sqrt{-q}, \sqrt{d})$ contains 1 and must be the unit ideal. Feeding this representation of 1 into the right side of (3.3) shows $\sqrt{-q}\mathcal{O}_E = \mathfrak{q}\mathcal{O}_E$.

The general classification theorem for (torsion-free) finitely generated modules over a Dedekind domain has the following form. Compare it with Theorem 3.1.

Theorem 3.3. *Let A be a Dedekind domain and M be a finitely generated torsion-free A-module. Then there is an $r \geq 1$ such that $M \cong A^{r-1} \oplus \mathfrak{a}$ as A-modules, where $\mathfrak{a}$ is an ideal of A. The ideal class of $\mathfrak{a}$ is well-defined by M.*

Proof. See [2, Prop. 24, Chapter VII]. It is the final proposition in the book. □

In particular, for a degree n extension of number fields E/F, there is an $\mathcal{O}_F$-module isomorphism $\mathcal{O}_E \cong \mathcal{O}_F^{n-1} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\mathcal{O}_F$ (possibly non-principal). Thus, in (1.1) it is possible to get a direct sum, but we must allow one of the x_i's to have coefficients running through an ideal of $\mathcal{O}_F$ rather than through $\mathcal{O}_F$ itself.

Finally, it is possible for $\mathcal{O}_E$ to be a free $\mathcal{O}_F$-module even if $\mathcal{O}_F$ is not a PID. For instance, let $F = \mathbf{Q}(\sqrt{-15})$ ($\mathcal{O}_F$ is not a PID) and $E = F(\sqrt{26}) = \mathbf{Q}(\sqrt{-15}, \sqrt{26})$. It can be shown that $\mathcal{O}_E = \mathcal{O}_F \oplus \mathcal{O}_F\sqrt{26}$.

References

[1] R. MacKenzie and J. Scheuneman, A Number Field Without a Relative Integral Basis, *Amer. Math. Monthly* **78** (1971), 882–883.

[2] N. Bourbaki, "Commutative Algebra," Addison-Wesley, 1972, Reading, MA.