# LOCAL COMPACTNESS OF CHARACTER GROUPS BY ASCOLI

## 1. Introduction

Let $G$ be a locally compact abelian group. (It is assumed, as usual, that all topological groups under discussion are Hausdorff.) A *character* of $G$ is a continuous homomorphism $G \to S^1$. We denote the set of characters of $G$ by $\widehat{G}$. It is an abelian group under pointwise multiplication.

We topologize $\widehat{G}$ with the subspace topology as a subset of the space $C(G, \mathbf{C})$ of continuous functions $G \to \mathbf{C}$ with the compact-open topology. That means basic open sets around the trivial character $\mathbf{1}$ in $\widehat{G}$ are

$$\{\chi \in \widehat{G} : |\chi - 1| < \varepsilon \text{ for all } x \in K\}$$

for compact $K$ in $G$ and $\varepsilon > 0$. The compact-open topology on $C(G, \mathbf{C})$ is Hausdorff, so the topology on $\widehat{G}$ is Hausdorff. With the above topology $\widehat{G}$ is a topological group and it is a closed subset of $C(G, \mathbf{C})$ (intuitively, a limit of homomorphisms is a homomorphism).

Our goal here is to explain why $\widehat{G}$ is locally compact. Local compactness is one of the first nontrivial properties of $\widehat{G}$, but in most books I looked at that cover harmonic analysis on general locally compact abelian groups, the local compactness is explained by properties of commutative Banach algebras (applied to $L^1(G)$) and then a comparison of two topologies on $\widehat{G}$: see [2, Theorem 3.2.1], [3, p. 86], [4, Theorem 23.15], [5, pp. 135, 137], and [8, p. 9, App. D4]. The only exceptions I found are [9, Sect. 27], whose first edition predates the use of Banach algebra techniques, and [7, Prop. 3.2(v)], which does not use Banach algebras but has a comparison of two topologies on $\widehat{G}$. The approach we take here doesn't use Banach algebras. It uses a standard theorem in analysis that describes when a set of continuous functions is compact: Ascoli's theorem. (The proof of Ascoli's theorem [6, Chap. 7, Theorem 6.1] involves showing in a few cases that two topologies on a set of functions are the same, so the "equal topologies" part of the standard proof of local compactness of $\widehat{G}$ is still present, but logically it is hidden away in Ascoli's theorem. The compactness in Ascoli's theorem relies on Tychonoff's theorem, and all proofs of local compactness of $\widehat{G}$ for general $G$ ultimately rely on Tychonoff's theorem in some form.)

## 2. Local compactness of $\widehat{G}$

We start with a standard lemma about continuity of a shift inside an $L^1$-function on $G$.

**Lemma 2.1.** *Fix $f \in L^1(G)$. For $y \in G$, let $L_y f : G \to \mathbf{C}$ by $(L_y f)(x) = f(yx)$. Then $L_y f \in L^1(G)$ and the map $y \mapsto L_y f$ from $G$ to $L^1(G)$ is continuous.*

*Proof.* By left invariance of Haar measure, $L_y f \in L^1(G)$ when $f \in L^1(G)$.

To prove continuity of $y \mapsto L_y f$, the basic idea is to check it directly on the dense subset $C_c(G) \subset L^1(G)$ and then extend it to all of $L^1(G)$ by an approximation argument.

It suffices to check continuity at the identity: for all $\varepsilon > 0$ there's a neighborhood $U_\varepsilon$ of $e$ such that $y \in U_\varepsilon \Rightarrow |L_y f - f|_1 < \varepsilon$. Indeed, if we have continuity at the identity, then for each $g \in G$ the set $U_\varepsilon g$ is a neighborhood of $g$ and

$$y \in U_\varepsilon g \Longrightarrow |L_y f - L_g f|_1 = \int_G |f(yx) - f(gx)|\, dx = \int_G |f(yg^{-1}x) - f(x)|\, dx = |L_{yg^{-1}} f - f|_1,$$

which is less than $\varepsilon$ since $yg^{-1} \in U_\varepsilon$. (This shows $y \mapsto L_y f$ is *uniformly* continuous in $y$.)

Reduction to the case $f \in C_c(G)$. Since $C_c(G)$ is dense in $L^1(G)$, for $\varepsilon > 0$ there is some $\varphi \in C_c(G)$ such that $|f - \varphi|_1 < \varepsilon/3$. Then for each $y \in G$,

$$|L_y f - f|_1 \le |L_y f - L_y \varphi|_1 + |L_y \varphi - \varphi|_1 + |\varphi - f|_1 = |f - \varphi|_1 + |L_y \varphi - \varphi|_1 + |\varphi - f|_1.$$

If the lemma is true for functions in $C_c(G)$ then $|L_y \varphi - \varphi|_1 < \varepsilon/3$ for all $y$ in some neighborhood of the identity in $G$, so $|L_y f - f|_1 < \varepsilon$ for all $y$ close to $e$.

Proof when $f \in C_c(G)$. Let $K$ be the support of $f$ and $C$ be a compact neighborhood of $e$. We will show for all $y$ in a suitable neighborhood of $e$ inside $C$ that $|L_y f - f|_1$ is arbitrarily small.

For $y \in C$, if $(L_y f)(x) - f(x) = f(yx) - f(x) \neq 0$ then $f(x) \neq 0$ or $f(yx) \neq 0$, so $x \in K$ or $yx \in K$, which means either way that $x \in C^{-1}K$ (note $e \in C^{-1}$). Both $K$ and $C^{-1}$ are compact (by continuity of inversion in $G$), so $C^{-1}K$ is compact (by continuity of multiplication in $G$). Thus $L_y f - f$ vanishes outside $C^{-1}K$, so

$$|L_y f - f|_1 = \int_G |f(yx) - f(x)|\, dx = \int_{C^{-1}K} |f(yx) - f(x)|\, dx \le \sup_{x \in G} |f(yx) - f(x)| \mu(C^{-1}K),$$

where $\mu$ is the Haar measure on $G$.

Functions in $C_c(G)$ are not just continuous but uniformly continuous: for all $\varepsilon > 0$ there's a neighborhood $V_\varepsilon$ of $e$ such that for all $g$ and $h$ in $G$, $gh^{-1} \in V_\varepsilon \Rightarrow |f(g) - f(h)| < \varepsilon$. The intersection $V_\varepsilon \cap C$ is a neighborhood of $e$ since $V_\varepsilon$ and $C$ are both neighborhoods of $e$. If $y \in V_\varepsilon \cap C$ then $(yx)x^{-1} \in V_\varepsilon$ for all $x \in G$, so $|f(yx) - f(x)| < \varepsilon$. Then from $y \in C$ we get

$$|L_y f - f|_1 \le \sup_{x \in G} |f(yx) - f(x)| \mu(C^{-1}K) \le \varepsilon \mu(C^{-1}K).$$

Make $\varepsilon$ arbitrarily small and we're done. □

**Remark 2.2.** Lemma 2.1 is true if $f \in L^p(G)$ for $1 \le p < \infty$, not just for $p = 1$.

**Lemma 2.3.** *For $\varepsilon > 0$, and $f \in L^1(G)$, $\{\chi \in \widehat{G} : |\int_G f(x)\chi(x)\, dx| \ge \varepsilon\}$ has compact closure in the compact-open topology of $\widehat{G}$.*

*Proof.* Set

$$C_{\varepsilon,f} := \left\{ \chi \in \widehat{G} : \left| \int_G f(x)\chi(x)\, dx \right| \ge \varepsilon \right\}.$$

To prove $C_{\varepsilon,f}$ has compact closure in $\widehat{G}$, recall that $\widehat{G}$ gets its topology from being a subset of $C(G, \mathbf{C})$ and $\widehat{G}$ is closed in $C(G, \mathbf{C})$. Therefore the closure of $C_{\varepsilon,f}$ in $\widehat{G}$ is the closure of $C_{\varepsilon,f}$ in $C(G, \mathbf{C})$ and a subset of $\widehat{G}$ is compact in $\widehat{G}$ if and only if it is compact in $C(G, \mathbf{C})$. Thus proving the lemma is equivalent to proving $C_{\varepsilon,f}$ has compact closure in $C(G, \mathbf{C})$.

For a locally compact Hausdorff space $X$ and a metric space $Y$, Ascoli's theorem [6, Chap. 7, Theorem 6.1] tells us necessary and sufficient conditions for a subset $\mathcal{F}$ of $C(X, Y)$ to have compact closure in the compact-open topology of $C(X, Y)$:

- for each $x \in X$, $\{\varphi(x) : \varphi \in \mathcal{F}\}$ has compact closure in $Y$,
- $\mathcal{F}$ is equicontinuous at each $x \in X$.

We will use this with $X = G$, $Y = \mathbf{C}$, and $\mathcal{F} = C_{\varepsilon,f}$. The first condition in Ascoli's theorem is automatic for $C_{\varepsilon,f}$ since characters have image in $S^1$ and all closed subsets of $S^1$ are compact.

We now prove the second condition, that $C_{\varepsilon,f}$ is equicontinuous at each point of $G$. Since characters on $G$ are homomorphisms to $S^1$, it suffices to show $C_{\varepsilon,f}$ is equicontinuous at the identity $e$ of $G$: for each $\delta > 0$ we want to find an open neighborhood $U_\delta$ of $e$ in $G$ such that

$$y \in U_\delta, \quad \chi \in C_{\varepsilon,f} \Longrightarrow |\chi(y) - 1| < \delta.$$

In other words, we want to turn the lower bound $|\int_G f(x)\chi(x)\,dx| \geq \varepsilon$ into an upper bound on $|\chi(y) - 1|$ (for all $y \in U_\delta$, where $U_\delta$ is not yet defined). The key idea is to get an upper bound on $|\chi(y)-1|$ where $y$ doesn't show up inside $\chi(y)$ anymore. For all $\chi \in \widehat{G}$ such that $|\int_G f(x)\chi(x)\,dx| \geq \varepsilon$ and all $y \in G$, we have

$$\begin{aligned}
|\chi(y) - 1|\varepsilon &\leq \left|(\chi(y) - 1)\int_G f(x)\chi(x)\,dx\right| \\
&= \left|\int_G f(x)\chi(xy)\,dx - \int_G f(x)\chi(x)\,dx\right| \\
&= \left|\int_G f(xy^{-1})\chi(x)\,dx - \int_G f(x)\chi(x)\,dx\right| \\
&= \left|\int_G (f(xy^{-1}) - f(x))\chi(x)\,dx\right| \\
&\leq \int_G \left|f(xy^{-1}) - f(x)\right|\,dx \quad \text{since } \chi(G) \subset S^1 \\
&= |L_{y^{-1}}f - f|_1,
\end{aligned}$$

so

$$|\chi(y) - 1| \leq \frac{1}{\varepsilon}|L_{y^{-1}}f - f|_1$$

for all $y \in G$. From continuity of $y \mapsto L_y f$ as a mapping $G \to L^1(G)$ (Lemma 2.1) and continuity of inversion on $G$, $|L_{y^{-1}}f - f|_1 \to 0$ as $y \to e$ in $G$. Therefore $|\chi(y) - 1| < \delta$ for all $y$ near $e$, and that level of nearness to $e$ gives us the desired set $U_\delta$. □

**Remark 2.4.** When $G = \mathbf{R}$, so $\widehat{G} \cong \mathbf{R}$, this lemma is essentially the Riemann–Lebesgue lemma: if $f \in L^1(\mathbf{R})$ then $\int_{\mathbf{R}} f(x)e^{2\pi ixy}\,dx \to 0$ as $|y| \to \infty$. We have $|\int_{\mathbf{R}} f(x)e^{2\pi ixy}\,dx| < \varepsilon$ by Lemma 2.3 when $|y|$ is large (outside the complement of a compact subset of $\mathbf{R}$, which are the closed and bounded subsets of $\mathbf{R}$).

**Theorem 2.5.** *When $G$ is a locally compact abelian group, the group $\widehat{G}$ is locally compact in the compact-open topology.*

*Proof.* Since $\widehat{G}$ is a topological group, it suffices to show the trivial character $\mathbf{1}$ has a basis of open neighborhoods with compact closure. Every neighborhood of $\mathbf{1}$ in $\widehat{G}$ contains some $N_{\mathbf{1}}(K, \varepsilon) = \{\chi \in \widehat{G} : |\chi(x) - 1| < \varepsilon \text{ for all } x \in K\}$, where $K$ is a nonempty compact subset of $G$ and $\varepsilon > 0$. Making $\varepsilon$ smaller makes $N_{\mathbf{1}}(K, \varepsilon)$ smaller, and making $K$ *larger* makes $N_{\mathbf{1}}(K, \varepsilon)$ smaller, so it suffices to show $N_{\mathbf{1}}(K, \varepsilon)$ has compact closure for all small enough $\varepsilon > 0$ and all large enough compact $K$. In particular, we can assume $\varepsilon \in (0, 1)$ and $K$ has positive Haar measure since every compact subset of $G$ with measure 0 is contained in a compact subset of $G$ with positive measure: $G$ has nonzero Haar measure, so inner regularity of Haar measure implies there's a compact subset $S$ with positive measure. Then if $\mu(K) = 0$, $K \cup S$ is compact with positive measure and contains $K$.

If $0 < \varepsilon < 1$ and $K$ is a compact subset of $G$ with positive Haar measure, we will show $N_{\mathbf{1}}(K, \varepsilon)$ has compact closure in $\widehat{G}$. Our argument is based on the compactness result in [4, Cor. 23.16].[1] The characteristic function $\xi_K$ is in $L^1(G)$ because compact subsets of $G$ have finite Haar measure. For all $\chi \in \widehat{G}$,

$$\begin{aligned}\mu(K) &= \int_G \xi_K(x)\,dx \\ &= \int_G \xi_K(x)\cdot(1-\chi(x))\,dx + \int_G \xi_K(x)\chi(x)\,dx \\ &= \int_K (1-\chi(x))\,dx + \int_G \xi_K(x)\chi(x)\,dx.\end{aligned}$$

Taking absolute values, if $\chi \in N_{\mathbf{1}}(K, \varepsilon)$ then

$$\mu(K) \le \int_K |1-\chi(x)|\,dx + \left|\int_G \xi_K(x)\chi(x)\,dx\right| \le \mu(K)\varepsilon + \left|\int_G \xi_K(x)\chi(x)\,dx\right|,$$

so

$$\left|\int_G \xi_K(x)\chi(x)\,dx\right| \ge (1-\varepsilon)\mu(K) > 0.$$

By Lemma 2.3, the set of $\chi \in \widehat{G}$ satisfying the above inequality (for fixed $K$ and $\varepsilon$) has compact closure in $\widehat{G}$. We have shown $N_{\mathbf{1}}(K, \varepsilon)$ is inside this set, so $N_{\mathbf{1}}(K, \varepsilon)$ also has compact closure in $\widehat{G}$. □

**Remark 2.6.** The set of continuous homomorphisms $G \to \mathbf{C}^\times$, like $\widehat{G}$, is an abelian group under pointwise multiplication. Denote it by $X(G)$. For example, $\widehat{\mathbf{Z}} \cong S^1$ (all maps $n \mapsto z^n$ for $z \in S^1$) and $X(\mathbf{Z}) \cong \mathbf{C}^\times$ (all maps $n \mapsto z^n$ for $z \in \mathbf{C}^\times$), where $\mathbf{Z}$ has the discrete topology. The group $X(G)$ is a topological group in the compact-open topology from $C(G, \mathbf{C})$. The proof of that involves a little more care than for $\widehat{G}$ since $|\chi(g)^{-1}| \neq |\chi(g)|$ if $\chi(g) \notin S^1$.

Unlike $\widehat{G}$, when $G$ is locally compact $X(G)$ is *not* always locally compact. For example, $X(\bigoplus_{k\ge1}\mathbf{Z}) \cong \prod_{k\ge1}\mathbf{C}^\times$, where $\prod_{k\ge1}\mathbf{C}^\times$ has the product topology [1, Example 2.4].

## References


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[1] This result was the starting point for developing this proof, and ironically it appears in [4] as a *corollary* to a proof of local compactness of $\widehat{G}$.