1. Introduction

Let $G$ be a locally compact abelian group. (It is assumed, as usual, that all topological groups under discussion are Hausdorff.) A character of $G$ is a continuous homomorphism $G \to S^1$. We denote the set of characters of $G$ by $\hat{G}$. It is an abelian group under pointwise multiplication.

We topologize $\hat{G}$ with the subspace topology as a subset of the space $C(G, \mathbb{C})$ of continuous functions $G \to \mathbb{C}$ with the compact-open topology. That means basic open sets around the trivial character $1$ in $\hat{G}$ are
\[ \{ \chi \in \hat{G} : |\chi - 1| < \varepsilon \text{ for all } x \in K \} \]
for compact $K$ in $G$ and $\varepsilon > 0$. The compact-open topology on $C(G, \mathbb{C})$ is Hausdorff, so the topology on $\hat{G}$ is Hausdorff. With the above topology $\hat{G}$ is a topological group and it is a closed subset of $C(G, \mathbb{C})$ (intuitively, a limit of homomorphisms is a homomorphism).

Our goal here is to explain why $\hat{G}$ is locally compact. Local compactness is one of the first nontrivial properties of $\hat{G}$, but in most books I looked at that cover harmonic analysis on general locally compact abelian groups, the local compactness is explained by properties of commutative Banach algebras (applied to $L^1(G)$) and then a comparison of two topologies on $\hat{G}$: see [2, Theorem 3.2.1], [3, p. 86], [4, Theorem 23.15], [5, pp. 135, 137], and [8, p. 9, App. D4]. The only exceptions I found are [9, Sect. 27], whose first edition predates the use of Banach algebra techniques, and [7, Prop. 3.2(v)], which does not use Banach algebras but has a comparison of two topologies on $\hat{G}$. The approach we take here doesn’t use Banach algebras. It uses a standard theorem in analysis that describes when a set of continuous functions is compact: Ascoli’s theorem. (The proof of Ascoli’s theorem [6, Chap. 7, Theorem 6.1] involves showing in a few cases that two topologies on a set of functions are the same, so the “equal topologies” part of the standard proof of local compactness of $\hat{G}$ is still present, but logically it is hidden away in Ascoli’s theorem. The compactness in Ascoli’s theorem relies on Tychonoff’s theorem, and all proofs of local compactness of $\hat{G}$ for general $G$ ultimately rely on Tychonoff’s theorem in some form.)

2. Local compactness of $\hat{G}$

We start with a standard lemma about continuity of a shift inside an $L^1$-function on $G$.

**Lemma 2.1.** Fix $f \in L^1(G)$. For $y \in G$, let $L_y f : G \to \mathbb{C}$ by $(L_y f)(x) = f(\text{yx})$. Then $L_y f \in L^1(G)$ and the map $y \mapsto L_y f$ from $G$ to $L^1(G)$ is continuous.

**Proof.** By left invariance of Haar measure, $L_y f \in L^1(G)$ when $f \in L^1(G)$.

To prove continuity of $y \mapsto L_y f$, the basic idea is to check it directly on the dense subset $C_c(G) \subset L^1(G)$ and then extend it to all of $L^1(G)$ by an approximation argument.

It suffices to check continuity at the identity: for all $\varepsilon > 0$ there’s a neighborhood $U_\varepsilon$ of $e$ such that $y \in U_\varepsilon \Rightarrow |L_y f - f|_1 < \varepsilon$. Indeed, if we have continuity at the identity, then for
each \( g \in G \) the set \( U_{\varepsilon}g \) is a neighborhood of \( g \) and

\[
y \in U_{\varepsilon}g \implies |Lyf - Lgf|_1 = \int_G |f(yx) - f(gx)| \, dx = \int_G |f(yg^{-1}x) - f(x)| \, dx = |L_{yg^{-1}}f - f|_1,
\]

which is less than \( \varepsilon \) since \( yg^{-1} \in U_{\varepsilon} \). (This shows \( y \mapsto L_yf \) is uniformly continuous in \( y \).)

Reduction to the case \( f \in C_c(G) \). Since \( C_c(G) \) is dense in \( L^1(G) \), for \( \varepsilon > 0 \) there is some \( \varphi \in C_c(G) \) such that \( |f - \varphi|_1 < \varepsilon/3 \). Then for each \( y \in G \),

\[
|L_yf - f|_1 \leq |L_yf - Ly\varphi|_1 + |Ly\varphi - \varphi|_1 + |\varphi - f|_1 = |f - \varphi|_1 + |Ly\varphi - \varphi|_1 + |\varphi - f|_1.
\]

If the lemma is true for functions in \( C_c(G) \) then \( |Ly\varphi - \varphi|_1 < \varepsilon/3 \) for all \( y \) in some neighborhood of the identity in \( G \), so \( |L_yf - f|_1 < \varepsilon \) for all \( y \) close to \( e \).

Proof when \( f \in C_c(G) \). Let \( K \) be the support of \( f \) and \( C \) be a compact neighborhood of \( e \). We will show for all \( y \) in a suitable neighborhood of \( e \) inside \( C \) that \( |L_yf - f|_1 \) is arbitrarily small.

For \( y \in C \), if \( (L_yf)(x) - f(x) = f(yx) - f(x) \neq 0 \) then \( f(x) \neq 0 \) or \( f(yx) \neq 0 \), so \( x \in K \) or \( yx \in K \), which means either way that \( x \in C^{-1}K \) (note \( e \in C^{-1} \)). Both \( K \) and \( C^{-1} \) are compact (by continuity of inversion in \( G \)), so \( C^{-1}K \) is compact (by continuity of multiplication in \( G \)). Thus \( L_yf - f \) vanishes outside \( C^{-1}K \), so

\[
|L_yf - f|_1 = \int_G |f(yx) - f(x)| \, dx = \int_{C^{-1}K} |f(yx) - f(x)| \, dx \leq \sup_{x \in G} |f(yx) - f(x)| \mu(C^{-1}K),
\]

where \( \mu \) is the Haar measure on \( G \).

Functions in \( C_c(G) \) are not just continuous but uniformly continuous: for all \( \varepsilon > 0 \) there’s a neighborhood \( V_{\varepsilon} \) of \( e \) such that for all \( g \) and \( h \) in \( G \), \( gh^{-1} \in V_{\varepsilon} \implies |f(g) - f(h)| < \varepsilon \). The intersection \( V_{\varepsilon} \cap C \) is a neighborhood of \( e \) since \( V_{\varepsilon} \) and \( C \) are both neighborhoods of \( e \). If \( y \in V_{\varepsilon} \cap C \) then \( (yx)x^{-1} \in V_{\varepsilon} \) for all \( x \in G \), so \( |f(yx) - f(x)| < \varepsilon \). Then from \( y \in C \) we get

\[
|L_yf - f|_1 \leq \sup_{x \in G} |f(yx) - f(x)| \mu(C^{-1}K) \leq \varepsilon \mu(C^{-1}K).
\]

Make \( \varepsilon \) arbitrarily small and we’re done. \( \square \)

**Remark 2.2.** Lemma 2.1 is true if \( f \in L^p(G) \) for \( 1 \leq p < \infty \), not just for \( p = 1 \).

**Lemma 2.3.** For \( \varepsilon > 0 \), and \( f \in L^1(G) \), \( \{ \chi \in \widehat{G} : |\int_G f(x)\chi(x) \, dx| \geq \varepsilon \} \) has compact closure in the compact-open topology of \( \widehat{G} \).

**Proof.** Set

\[
C_{\varepsilon,f} := \left\{ \chi \in \widehat{G} : \left| \int_G f(x)\chi(x) \, dx \right| \geq \varepsilon \right\}.
\]

To prove \( C_{\varepsilon,f} \) has compact closure in \( \widehat{G} \), recall that \( \widehat{G} \) gets its topology from being a subset of \( C(G,\mathbb{C}) \) and \( \widehat{G} \) is closed in \( C(G,\mathbb{C}) \). Therefore the closure of \( C_{\varepsilon,f} \) in \( \widehat{G} \) is the closure of \( C_{\varepsilon,f} \) in \( C(G,\mathbb{C}) \) and a subset of \( \widehat{G} \) is compact in \( \widehat{G} \) if and only if it is compact in \( C(G,\mathbb{C}) \). Thus proving the lemma is equivalent to proving \( C_{\varepsilon,f} \) has compact closure in \( C(G,\mathbb{C}) \).

For a locally compact Hausdorff space \( X \) and a metric space \( Y \), Ascoli’s theorem [6, Chap. 7, Theorem 6.1] tells us necessary and sufficient conditions for a subset \( \mathcal{F} \) of \( C(X,Y) \) to have compact closure in the compact-open topology of \( C(X,Y) \):

- for each \( x \in X \), \( \{ \varphi(x) : \varphi \in \mathcal{F} \} \) has compact closure in \( Y \),
- \( \mathcal{F} \) is equicontinuous at each \( x \in X \).
We will use this with $X = G$, $Y = C$, and $\mathcal{F} = C_{\varepsilon,f}$. The first condition in Ascoli’s theorem is automatic for $C_{\varepsilon,f}$ since characters have image in $S^1$ and all closed subsets of $S^1$ are compact.

We now prove the second condition, that $C_{\varepsilon,f}$ is equicontinuous at each point of $G$. Since characters on $G$ are homomorphisms to $S^1$, it suffices to show $C_{\varepsilon,f}$ is equicontinuous at the identity $e$ of $G$: for each $\delta > 0$ we want to find an open neighborhood $U_\delta$ of $e$ in $G$ such that

$$y \in U_\delta, \; \chi \in C_{\varepsilon,f} \implies |\chi(y) - 1| < \delta.$$ 

In other words, we want to turn the lower bound $|\int_G f(x)\chi(x)\,dx| \geq \varepsilon$ into an upper bound on $|\chi(y) - 1|$ (for all $y \in U_\delta$, where $U_\delta$ is not yet defined). The key idea is to get an upper bound on $|\chi(y) - 1|$ where $y$ doesn’t show up inside $\chi(y)$ anymore. For all $\chi \in \hat{G}$ such that $|\int_G f(x)\chi(x)\,dx| \geq \varepsilon$ and all $y \in G$, we have

$$|\chi(y) - 1|\varepsilon \leq |(\chi(y) - 1)\int_G f(x)\chi(x)\,dx|$$

$$= \left| \int_G f(x)\chi(xy)\,dx - \int_G f(x)\chi(x)\,dx \right|$$

$$= \left| \int_G f(xy^{-1})\chi(x)\,dx - \int_G f(x)\chi(x)\,dx \right|$$

$$= \left| \int_G (f(xy^{-1}) - f(x))\chi(x)\,dx \right|$$

$$\leq \int_G |f(xy^{-1}) - f(x)|\,dx \text{ since } \chi(G) \subset S^1$$

$$= |L_{y^{-1}}f - f|_1,$$

so

$$|\chi(y) - 1| \leq \frac{1}{\varepsilon}|L_{y^{-1}}f - f|_1$$

for all $y \in G$. From continuity of $y \mapsto L_yf$ as a mapping $G \to L^1(G)$ (Lemma 2.1) and continuity of inversion on $G$, $|L_{y^{-1}}f - f|_1 \to 0$ as $y \to e$ in $G$. Therefore $|\chi(y) - 1| < \delta$ for all $y$ near $e$, and that level of nearness to $e$ gives us the desired set $U_\delta$. 

**Remark 2.4.** When $G = \mathbb{R}$, so $\hat{G} \cong \mathbb{R}$, this lemma is essentially the Riemann–Lebesgue lemma: if $f \in L^1(\mathbb{R})$ then $\int_{\mathbb{R}} f(x)e^{2\pi i xy}\,dx \to 0$ as $|y| \to \infty$. We have $|\int_{\mathbb{R}} f(x)e^{2\pi i xy}\,dx| < \varepsilon$ by Lemma 2.3 when $|y|$ is large (outside the complement of a compact subset of $\mathbb{R}$, which are the closed and bounded subsets of $\mathbb{R}$).

**Theorem 2.5.** When $G$ is a locally compact abelian group, the group $\hat{G}$ is locally compact in the compact-open topology.

**Proof.** Since $\hat{G}$ is a topological group, it suffices to show the trivial character $1$ has a basis of open neighborhoods with compact closure. Every neighborhood of $1$ in $\hat{G}$ contains some $N_1(K, \varepsilon) = \{\chi \in \hat{G} : |\chi(x) - 1| < \varepsilon \text{ for all } x \in K\}$, where $K$ is a nonempty compact subset of $G$ and $\varepsilon > 0$. Making $\varepsilon$ smaller makes $N_1(K, \varepsilon)$ smaller, and making $K$ larger makes $N_1(K, \varepsilon)$ smaller, so it suffices to show $N_1(K, \varepsilon)$ has compact closure for all small enough $\varepsilon > 0$ and all large enough compact $K$. In particular, we can assume $\varepsilon \in (0,1)$ and $K$ has positive Haar measure since every compact subset of $G$ with measure $0$ is contained in a compact subset of $G$ with positive measure: $G$ has nonzero Haar measure, so inner
regularity of Haar measure implies there’s a compact subset $S$ with positive measure. Then if $\mu(K) = 0$, $K \cup S$ is compact with positive measure and contains $K$.

If $0 < \varepsilon < 1$ and $K$ is a compact subset of $G$ with positive Haar measure, we will show $N_1(K, \varepsilon)$ has compact closure in $\hat{G}$. Our argument is based on the compactness result in [4, Cor. 23.16]. The characteristic function $\xi_K$ is in $L^1(G)$ because compact subsets of $G$ have finite Haar measure. For all $\chi \in \hat{G}$,

$$
\mu(K) = \int_G \xi_K(x) \, dx
= \int_G \xi_K(x) \cdot (1 - \chi(x)) \, dx + \int_G \xi_K(x) \chi(x) \, dx
= \int_K (1 - \chi(x)) \, dx + \int_G \xi_K(x) \chi(x) \, dx.
$$

Taking absolute values, if $\chi \in N_1(K, \varepsilon)$ then

$$
\mu(K) \leq \int_K |1 - \chi(x)| \, dx + \left| \int_G \xi_K(x) \chi(x) \, dx \right| \leq \mu(K) \varepsilon + \left| \int_G \xi_K(x) \chi(x) \, dx \right|
$$

so

$$
\left| \int_G \xi_K(x) \chi(x) \, dx \right| \geq (1 - \varepsilon)\mu(K) > 0.
$$

By Lemma 2.3, the set of $\chi \in \hat{G}$ satisfying the above inequality (for fixed $K$ and $\varepsilon$) has compact closure in $\hat{G}$. We have shown $N_1(K, \varepsilon)$ is inside this set, so $N_1(K, \varepsilon)$ also has compact closure in $\hat{G}$.

**Remark 2.6.** The set of continuous homomorphisms $G \to \mathbb{C}^\times$, like $\hat{G}$, is an abelian group under pointwise multiplication. Denote it by $X(G)$. For example, $\hat{Z} \cong S^1$ (all maps $n \mapsto z^n$ for $z \in S^1$) and $X(\mathbb{Z}) \cong \mathbb{C}^\times$ (all maps $n \mapsto z^n$ for $z \in \mathbb{C}^\times$), where $\mathbb{Z}$ has the discrete topology. The group $X(G)$ is a topological group in the compact-open topology from $C(G, \mathbb{C})$. The proof of that involves a little more care than for $\hat{G}$ since $|\chi(g)^{-1}| \neq |\chi(g)|$ if $\chi(g) \notin S^1$.

Unlike $\hat{G}$, when $G$ is locally compact $X(G)$ is not always locally compact. For example, $X(\bigoplus_{k \geq 1} \mathbb{Z}) \cong \prod_{k \geq 1} \mathbb{C}^\times$, where $\prod_{k \geq 1} \mathbb{C}^\times$ has the product topology [1, Example 2.4].

**References**


---

¹This result was the starting point for developing this proof, and ironically it appears in [4] as a corollary to a proof of local compactness of $\hat{G}$.