# EQUIVALENCE OF ABSOLUTE VALUES

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## 1. Introduction

An absolute value $|\cdot|$ on a field $K$ defines a metric on $K$ by $d(x,y) = |x-y|$, from which we get open subsets and closed subsets of $K$ relative to $|\cdot|$. Different absolute values always define different metrics since the absolute value can be recovered from the metric it defines ($|x| = d(x,0)$), but different absolute values could define the same concept of open subset.

**Example 1.1.** On $\mathbf{R}$ let $|\cdot|$ be the usual absolute value. Another absolute value on $\mathbf{R}$ is $|x|' = \sqrt{|x|}$ (the triangle inequality for $|\cdot|'$ comes from $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$ for $a, b \geq 0$). The condition $|x-a|' < r$ is the same as $|x-a| < r^2$, so open balls in $\mathbf{R}$ defined by $|\cdot|$ and $|\cdot|'$ are the same (even if the radii don't match). Open subsets for an absolute value are unions of open balls for that absolute value, so the open subsets of $\mathbf{R}$ for $|\cdot|$ and $|\cdot|'$ coincide.

**Definition 1.2.** Two absolute values on a field $K$ are called *equivalent* if they define the same open subsets of $K$.

In Example 1.1, $|\cdot|$ and $\sqrt{|\cdot|}$ on $\mathbf{R}$ are equivalent absolute values. In Section 2 we will show equivalent absolute values are always related in a similar way to the absolute values in the example: one is a power of the other. In Section 3 we will show inequivalent (nontrivial) absolute values behave independently as far as convergence is concerned.

## 2. Equivalent absolute values and powers

**Theorem 2.1.** *Let $K$ be a field and $|\cdot|$ and $|\cdot|'$ be two absolute values on $K$. They are equivalent if and only if $|\cdot|' = |\cdot|^t$ for some $t > 0$.*

**Remark 2.2.** This theorem is *not* saying that if $|\cdot|$ is an absolute value then $|\cdot|^t$ is an absolute value for all $t > 0$. If $0 < t \leq 1$ then $|\cdot|^t$ is an absolute value, but if $t > 1$ then $|\cdot|^t$ might not satisfy the triangle inequality (*e.g.*, if $|\cdot|$ is the usual absolute value on $\mathbf{R}$).

*Proof.* The proof is easy if $|\cdot|$ or $|\cdot|'$ is trivial: for the trivial absolute value on $K$ all subsets of $K$ are open, and this is not true otherwise (one-element subsets of $K$ are not open for a nontrivial absolute value), so the trivial absolute value on $K$ is equivalent only to itself.

From now on let $|\cdot|$ and $|\cdot|'$ be nontrivial absolute values on $K$.

($\Longleftarrow$) If $|\cdot|' = |\cdot|^t$ then the open balls in $K$ for $|\cdot|$ and $|\cdot|'$ are the same (even if not for the same radii):

$$\{x : |x-a| < r\} = \{x : |x-a|' < r^t\}, \quad \{x : |x-a|' < r\} = \{x : |x-a| < r^{1/t}\}. \tag{2.1}$$

As in Example 1.1, the open subsets of $K$ for $|\cdot|$ and $|\cdot|'$ are the same since open subsets for an absolute value are the unions of open balls for that absolute value.

($\Longrightarrow$) Since we are assuming the open subsets of $K$ defined by $|\cdot|$ and $|\cdot|'$ are the same, we can use the term "open subset" unambiguously for both absolute values. However, we can't yet say the open *balls* for one absolute value are also open balls for the other absolute value (but it will turn out to be so after the theorem is proved, by (2.1)). All we can be sure of is that an open ball for one absolute value is an open subset for the other absolute value.

First we will show that $|x| < 1 \iff |x|' < 1$. This is based on the following incompatible consequences of the conditions $|x| < 1$ and $|x| \geq 1$.

(1) If $|x| < 1$ then $|x^n| \to 0$, so for all open subsets $U$ of $K$ containing 0 there is an $N \geq 1$ such that $n \geq N \implies x^n \in U$.
(2) If $|x| \geq 1$ then $|x^n| \geq 1$ for all $n$, so there is an open subset $U$ of $K$ containing 0 (namely $\{y : |y| < 1\}$) such that $x^n \notin U$ for all $n$.

These are true with $|\cdot|'$ in place of $|\cdot|$, so the conditions $|x| < 1$ and $|x|' < 1$ are each equivalent to saying for every open subset $U$ containing 0 that $x^n \in U$ for all but finitely many $n$, so $|x| < 1 \iff |x|' < 1$.

The rest of the proof will follow from $|x| < 1 \iff |x|' < 1$. First, $|x| > 1 \iff |x|' > 1$:

$$|x| > 1 \iff \left|\frac{1}{x}\right| < 1 \iff \left|\frac{1}{x}\right|' < 1 \iff |x|' > 1. \tag{2.2}$$

Thus

$$|x| = 1 \iff |x| \not< 1 \text{ and } |x| \not> 1 \iff |x|' \not< 1 \text{ and } |x|' \not> 1 \iff |x|' = 1. \tag{2.3}$$

To prove there is $t > 0$ such that $|x|' = |x|^t$ for all $x \in K$ we can assume $x \neq 0$ and also $|x| \neq 1$ since $|x| = 1 \iff |x|' = 1$ by (2.3) (a value for $t$ doesn't matter). It suffices to prove $|x|' = |x|^t$ when $|x| > 1$ (replace $x$ with $1/x$ to settle the case $0 < |x| < 1$). To have $|x|' = |x|^t$ for some $t > 0$ and all $x \in K$ with $|x| > 1$ means $t = \log|x|'/\log|x|$ when $|x| > 1$, so we want to prove the ratio

$$\frac{\log|x|'}{\log|x|}$$

is the same for all $x$ such that $|x| > 1$. The common value of this ratio would be $t$.

Let $x$ and $y$ in $K$ satisfy $|x| > 1$ and $|y| > 1$. Then also $|x|' > 1$ and $|y|' > 1$ by (2.2). To prove $\log|x|'/\log|x| = \log|y|'/\log|y|$, assume these two real numbers are not equal. Then without loss of generality $\log|x|'/\log|x| < \log|y|'/\log|y|$. The logarithms are all positive, so $\log|x|'/\log|y|' < \log|x|/\log|y|$. There is a rational number lying in between:

$$\frac{\log|x|'}{\log|y|'} < \frac{m}{n} < \frac{\log|x|}{\log|y|}$$

for some positive integers $m$ and $n$. Then

$$\frac{\log|x|'}{\log|y|'} < \frac{m}{n} \implies n\log|x|' < m\log|y|' \implies |x^n|' < |y^m|' \implies \left|\frac{x^n}{y^m}\right|' < 1$$

and

$$\frac{m}{n} < \frac{\log|x|}{\log|y|} \implies m\log|y| < n\log|x| \implies |y^m| < |x^n| \implies \left|\frac{x^n}{y^m}\right| > 1,$$

and this contradicts the equivalence of $|z| < 1$ with $|z|' < 1$ for all $z \in K$. $\square$

**Corollary 2.3.** *If $|\cdot|$ and $|\cdot|'$ are nontrivial absolute values on a field $K$ and $|x| < 1 \implies |x|' < 1$ then the absolute values are equivalent.*

*Proof.* We will prove $|x|' < 1 \Longrightarrow |x| < 1$. Thus $|x| < 1 \Longleftrightarrow |x|' < 1$, and the part of the proof of Theorem 2.1 that follows from this, starting from (2.2), implies $|\cdot|' = |\cdot|^t$ for some $t > 0$, so $|\cdot|$ and $|\cdot|'$ are equivalent.

We argue by contradiction. Assume it is not true that $|x|' < 1 \Longrightarrow |x| < 1$ in general. Then there is an $x_0 \in K$ such that $|x_0|' < 1$ and $|x_0| \geq 1$. Since $|\cdot|$ is nontrivial, there is an $a \in K$ such that $0 < |a| < 1$. From $|1/x_0| \leq 1$ we get $|1/x_0^n| \leq 1$ for all $n \geq 1$, so

$$\left|\frac{a}{x_0^n}\right| \leq |a| < 1 \Longrightarrow \left|\frac{a}{x_0^n}\right|' < 1$$

since, by hypothesis, always $|x| < 1 \Longrightarrow |x|' < 1$. Clearing denominators,

$$|a|' < |x_0^n|' \tag{2.4}$$

for all $n$. Since $|x_0|' < 1$, we have $|x_0^n|' \to 0$, but this contradicts (2.4) for large enough $n$. Therefore there is no such $x_0$, so $|x|' < 1 \Longrightarrow |x| < 1$. □

**Corollary 2.4.** *If $|\cdot|$ and $|\cdot|'$ are nontrivial absolute values on a field $K$ and $|x| \leq 1 \Longrightarrow |x|' \leq 1$ then the absolute values are equivalent.*

*Proof.* By Corollary 2.3, it suffices to prove $|x| < 1 \Longrightarrow |x|' < 1$ for $x \in K$.

We have by hypothesis $|x| < 1 \Longrightarrow |x|' \leq 1$, so we want to show there is no $x \in K$ such that $|x| < 1$ and $|x|' = 1$. Assume some $x_0 \in K$ satisfies $|x_0| < 1$ and $|x_0|' = 1$. For every $a \in K$, from $|x_0| < 1$ we get $|ax_0^n| \to 0$, so $|ax_0^n| < 1$ for all large $n$. Then $|ax_0^n|' \leq 1$ for all large $n$. Since $|x_0|' = 1$ we have $|a|' \leq 1$, so all elements of $K^\times$ have absolute value at most 1. This implies $|\cdot|'$ is trivial, since otherwise some $|a|' > 1$ for some $a$. We were assuming from the start that $|\cdot|'$ is nontrivial, so we have a contradiction. Thus the condition $|x| < 1$ must always imply $|x|' < 1$. □

**Corollary 2.5.** *If $|\cdot|$ and $|\cdot|'$ are nontrivial absolute values on a field $K$ and $|x| < 1 \Longrightarrow |x|' \leq 1$ then the absolute values are equivalent.*

*Proof.* The proof of Corollary 2.4 works here *verbatim*, since all we used in that proof was $|x| < 1 \Longrightarrow |x|' \leq 1$, not the stated hypothesis $|x| \leq 1 \Longrightarrow |x|' \leq 1$. □

## 3. Independent approximations with inequivalent absolute values

Two equivalent absolute values define the same concept of convergence and limit: if $|\cdot|$ and $|\cdot|'$ are equivalent then $|\cdot|' = |\cdot|^t$ by Theorem 2.1, so $|x_n - x| \to 0$ is the same as $|x_n - x|' \to 0$. (A proof of this without Theorem 2.1 is in the appendix.) In this section we show every *finite* set of inequivalent nontrivial absolute values on $K$ has no relations among the notions of limit that each of the absolute values defines on $K$.

**Lemma 3.1.** *If $|\cdot|_1, \ldots, |\cdot|_n$ are a finite list of inequivalent nontrivial absolute values on a field $K$, where $n \geq 2$, then there is an $x \in K$ such that $|x|_1 > 1$ and $|x|_i < 1$ for $i \neq 1$.*

*Proof.* We use induction on $n$.

The base case $n = 2$ follows from Corollary 2.5: if there were no $x$ such that $|x|_1 > 1$ and $|x|_2 < 1$, then $|x|_2 < 1 \Longrightarrow |x|_1 \leq 1$ for all $x$ in $K$, and that implies $|\cdot|_1$ and $|\cdot|_2$ are equivalent by Corollary 2.5.

Now take $n \geq 3$ and assume the lemma is proved for all sets of $n - 1$ inequivalent nontrivial absolute values on $K$. By induction there is $y \in K$ such that $|y|_1 > 1$ and $|y|_i < 1$ for $i = 2, \ldots, n-1$, and by the base case there is $z \in K$ such that $|z|_1 > 1$ and $|z|_n < 1$. Neither $y$ nor $z$ is 0 since they have some absolute value greater than 1.

Case 1: $|y|_n \le 1$. We can use $x = y^k z$ for large enough $k$:

$$|x|_i = |y|_i^k |z|_i,$$

so for large $k$ we have $|x|_1 > 1$ (because $|y|_1 > 1$, regardless of the value of $|z|_1$) and $|x|_i < 1$ for $i = 2, \dots, n-1$ (because $|y|_i < 1$, regardless of the value of $|z|_i$). At $i = n$, since $|y|_n \le 1$ we have $|x|_n \le |z|_n < 1$.

Case 2: $|y|_n > 1$. The ratio $y^k/(1 + y^k)$ has different limiting behavior relative to an absolute value $|\cdot|$ depending on whether $|y| < 1$ or $|y| > 1$:

$$|y| < 1 \Longrightarrow y^k \to 0 \Longrightarrow \frac{y^k}{1+y^k} \to \frac{0}{1+0} = 0,$$

and

$$|y| > 1 \Longrightarrow \frac{1}{y^k} \to 0 \Longrightarrow \frac{y^k}{1+y^k} = \frac{1}{1/y^k + 1} \to \frac{1}{0+1} = 1,$$

where the convergence in both cases is relative to $|\cdot|$. That is, if $|y| < 1$ then $|y^k/(1+y^k)| \to 0$ and if $|y| > 1$ then $|y^k/(1+y^k) - 1| \to 0$. (If $|y| = 1$ we can't say anything about the behavior of $y^k/(1 + y^k)$, but this won't matter for us because $|y|_i \ne 1$ for all $i$ from 1 to $n$.)

Since $|y|_1 > 1$ and $|y|_n > 1$ we have

$$\left|\frac{y^k}{1+y^k} - 1\right|_1 \to 0, \quad \left|\frac{y^k}{1+y^k} - 1\right|_n \to 0$$

as $k \to \infty$. Since $|y|_i < 1$ for $2 \le i \le n-1$ we have

$$\left|\frac{y^k}{1+y^k}\right|_i \to 0$$

for $2 \le i \le n-1$.

For $k \ge 1$ set

$$x_k = \frac{y^k}{1+y^k} z.$$

We will show that we can use $x = x_k$ for a sufficiently large $k$. If $i = 1$ or $n$ then relative to $|\cdot|_i$ we have $y^k/(1 + y^k) \to 1$, so $x_k \to z$ and thus $|x_k|_i \to |z|_i$. Since $|z|_1 > 1$ and $|z|_n < 1$, we get $|x_k|_1 > 1$ and $|x_k|_n < 1$ for large $k$. For $2 \le i \le n-1$, relative to $|\cdot|_i$ we have $y^k/(1 + y^k) \to 0$, so $x_k \to 0$ and thus $|x_k|_i < 1$ for large $k$. □

**Remark 3.2.** Lemma 3.1 does *not* extend to infinite sets of inequivalent nontrivial absolute values. For example, on $\mathbf{Q}$ with the classical absolute value $|\cdot|_\infty$ and the $p$-adic absolute values $|\cdot|_p$ there is no $x \in \mathbf{Q}$ satisfying $|x|_\infty > 1$ and $|x|_p < 1$ for all primes $p$. Having $|x|_p < 1$ for all primes $p$ means the reduced form numerator of $x$ is divisible by every prime number, so $x = 0$, but then it is not true that $|x|_\infty > 1$.

**Theorem 3.3.** *If $|\cdot|_1, \dots, |\cdot|_n$ are inequivalent nontrivial absolute values on a field $K$, where $n \ge 2$, then for all choices of $x_1, \dots, x_n$ in $K$ and $\varepsilon > 0$ there is an $x \in K$ such that*

$$|x - x_i|_i < \varepsilon$$

*for $i = 1, \dots, n$.*

*Proof.* First we will treat the special case that $(x_1, x_2, \dots, x_n) = (1, 0, \dots, 0)$. We seek $c \in K$ such that $|c - 1|_1 < \varepsilon$ and $|c|_i < \varepsilon$ for $i = 2, \dots, n$. By Lemma 3.1 there is an $a \in K$ such that $|a|_1 > 1$ and $|a|_i < 1$ for $i \ne 1$. Then $a^k/(1 + a^k) = 1/(1/a^k + 1) \to 1$ relative to $|\cdot|_1$ since $|a|_1 > 1$ and $a^k/(1+a^k) \to 0$ relative to $|\cdot|_i$ for $i \neq 1$ since $|a|_i < 1$, so for large $k$ the number $c = a^k/(1+a^k)$ in $K$ satisfies $|c-1|_1 < \varepsilon$ and $|c|_i < \varepsilon$ for $i = 2, \ldots, n$. Permuting the roles of the different absolute values, we get a similar result with $|\cdot|_1$ replaced by the other absolute values in the list.

For a general $n$-tuple $(x_1, \ldots, x_n)$ in $K^n$, and each $i = 1, \ldots, n$, there is $c_i \in K$ such that $|c_i - 1|_i$ is very small and $|c_i|_j$ is very small for all $j \neq i$. Define the linear combination

$$x := c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = \sum_{j=1}^{n} c_j x_j.$$

This has coefficient $c_i$ close to 1 relative to $|\cdot|_i$ and coefficients $c_j$ close to 0 relative to $|\cdot|_i$ for $j \neq i$, so relative to $|\cdot|_i$ the linear combination $x$ above is close to

$$\sum_{j \neq i} 0 \cdot x_j + 1 \cdot x_i = x_i$$

relative to $|\cdot|_i$, which is what we want for the theorem.

More explicitly, for each $i$

$$|x - x_i|_i = \left| \sum_{j \neq i} c_j x_j + (c_i - 1)x_i \right|_i \leq \sum_{j \neq i} |c_j|_i |x_j|_i + |c_i - 1|_i |x_i|_i,$$

so if we set $s > \max_{i,j} |x_i|_j$ and we pick each $c_i$ in $K$ so that $|c_i - 1|_i \leq \varepsilon/(ns)$ and $|c_i|_j \leq \varepsilon/(ns)$ for $j \neq i$, then for each $i$

$$|x - x_i|_i \leq \sum_{j \neq i} \frac{\varepsilon}{ns} |x_j|_i + \frac{\varepsilon}{ns} |x_i|_i \leq \frac{\varepsilon}{ns}(|x_1|_i + \cdots + |x_n|_i) < \frac{\varepsilon}{ns}(ns) = \varepsilon.$$

□

## Appendix A. Equivalence in terms of convergence

**Theorem A.1.** *The following properties of absolute values $|\cdot|$ and $|\cdot|'$ on a field $K$ are equivalent to each other.*

1. *The open subsets of $K$ defined by $|\cdot|$ and $|\cdot|'$ are the same.*
2. *The convergent subsequences and their limits in $K$ defined by $|\cdot|$ and $|\cdot|'$ are the same.*

*Proof.* (1) $\Longrightarrow$ (2): Convergence in $K$ relative to an absolute value can be described in terms of open subsets of $K$ for that absolute value: to say $|x_n - x| \to 0$ in $K$ means for every open subset $U \subset K$ relative to $|\cdot|$ such that $x \in U$, there is an $N \geq 1$ such that $n \geq N \Longrightarrow x_n \in U$. By (1), $U$ being open relative to $|\cdot|$ is also open relative to $|\cdot|'$, so for every open subset $U \subset K$ relative to $|\cdot|'$ such that $x \in U$, there is an $N \geq 1$ such that $n \geq N \Longrightarrow x_n \in U$. Thus $|x_n - x|' \to 0$.

(2) $\Longrightarrow$ (1): Since open subsets are complements of closed subsets, proving (1) is equivalent to proving the closed subsets of $K$ defined by $|\cdot|$ and $|\cdot|'$ are the same. Let $C \subset K$ be closed relative to $|\cdot|$. To show $C$ is closed relative to $|\cdot|'$, let $\{x_n\}$ be a sequence in $C$ that converges to some $x \in K$ relative to $|\cdot|'$: $|x_n - x|' \to 0$. By hypothesis, $|x_n - x| \to 0$, so $x \in C$ since $C$ is closed relative to $|\cdot|$. Hence every sequence in $C$ that converges in $K$ relative to $|\cdot|'$ has its limit in $C$, so $C$ is closed relative to $|\cdot|'$.

The proof that every closed subset of $K$ relative to $|\cdot|'$ is closed relative to $|\cdot|$ is the same, with the roles of $|\cdot|$ and $|\cdot|'$ exchanged. □

We can also prove (1) implies (2) by writing $|\cdot|' = |\cdot|^t$, from Theorem 2.1, since then it is obvious that $|x_n - x| \to 0$ means the same thing as $|x_n - x|' \to 0$. We did not use that in the proof in order to show that the equivalence of (1) and (2) in Theorem A.1 does not depend on explicit formulas relating $|\cdot|$ and $|\cdot|'$.