HISTORY OF CLASS FIELD THEORY

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1. Introduction

Class field theory is the description of abelian extensions of global fields and local fields. The label "class field" refers to a field extension satisfying a technical property that is historically related to ideal class groups, and one of the main theorems is that class fields are the same as abelian extensions.

Three themes in number theory at the end of the 19th century led to class field theory: relations between abelian extensions and ideal class groups, density theorems for primes (and L-functions), and reciprocity laws. We will outline how class field theory developed from these initial ideas through the work of Kronecker, Weber, Hilbert, Takagi, Artin, Hasse, and Chevalley. One point concerning chronology: while we usually attribute results to mathematicians in years according to the appearance of the published papers, the actual work was often done earlier (*e.g.*, Takagi's fundamental paper in 1920 was based on work he had carried out several years earlier, with its publishing outside of Japan being delayed by World War I).

Some general surveys on the development of class field theory are [4], [5] (a more detailed version of [4], but not as accessible), [7], [9], [10], and the beginning of Part 2 of [11]. At the start of each section, particular references for that material are indicated. References to original papers are not given here, but can be found by consulting the cited sources.

Concerning notation, generally L/K will be an extension of number fields, with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$, and E/F will be an extension of local fields. The set of primes in K that split completely in L is $\mathrm{Spl}(L/K)$. Given a place v on K and a place w lying over it in L, $D(w|v)$ and $I(w|v)$ are the associated decomposition and inertia groups.[1]

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2. Beginnings (Kronecker)

References: [13], [15], [21].

In 1853, Kronecker announced what is now called the Kronecker–Weber theorem.

Theorem 2.1 (Kronecker–Weber)**.** *Every finite abelian extension of* $\mathbf{Q}$ *lies in a cyclotomic field* $\mathbf{Q}(\zeta_m)$ *for some* m*.*

Kronecker's proof, by his own admittance, had difficulties with extensions of 2-power degree. The first accepted proof was by Weber in 1886, but it also had an error at 2 that went unnoticed for about 90 years. The first correct proof was Hilbert's in 1896. It's

[1] In the literature these are also denoted $Z(w|v)$ and $T(w|v)$ from the German: Zerlegungsgruppe and Trägheitsgruppe.

worth saying something about the strategy of the proof because of its relation to Hilbert's later ideas on class field theory. Hilbert starts with an abelian extension $L/\mathbf{Q}$ and uses his recently developed theory of higher ramification groups to show L lies in a succession of fields of the form $F_n(\zeta_n)$ where F_n is a subfield of L (so F_n is necessarily abelian over $\mathbf{Q}$) such that the ramification in $F_n/\mathbf{Q}$ can be made smaller in exchange for adjoining appropriate roots of unity. Eventually F_n is an abelian unramified extension of $\mathbf{Q}$, so $F_n = \mathbf{Q}$ since $\mathbf{Q}$ has no proper unramified extensions (abelian or not). At this point we have $L \subset \mathbf{Q}(\zeta_n)$ and the proof is complete.

Abelian extensions of $\mathbf{Q}(i)$ were constructed by Abel (1829) with special values of the lemniscatic sine function $\mathrm{sl}(z)$, whose period lattice is essentially $\mathbf{Z}[i]$. (Abel was following up on suggestions of Gauss in the *Disquisitones Arithmeticae* that there is a theory of arc division on the lemniscate that parallels the theory of arc division on the circle using roots of unity.) Extending Abel's work, Kronecker was able to generate abelian extensions of imaginary quadratic fields using special values of elliptic and modular functions. In a letter to Dedekind in 1880, Kronecker's described his "Jugendtraum" (dream of youth)[2] as the hope that every finite abelian extension of an imaginary quadratic field lies in one of the extensions he had found. As a particular example, he expected that every finite abelian extension of $\mathbf{Q}(i)$ lies in a field $\mathbf{Q}(i, \mathrm{sl}(\omega/m))$, where $\omega \approx 2.622$ is the lemniscatic analogue of π. This is similar to the Kronecker-Weber theorem, with $\mathrm{sl}(\omega/m)$ analogous to $\zeta_m = e^{2\pi i/m}$.

An important case of Kronecker's work uses the j-function: if K is imaginary quadratic and we write $\mathcal{O}_K = \mathbf{Z} + \mathbf{Z}\tau_1$, where τ_1 is in the upper half-plane, Kronecker showed $j(\tau_1)$ is algebraic over K and its K-conjugates are $j(\tau_1), \ldots, j(\tau_h)$ where the lattices $\mathbf{Z} + \mathbf{Z}\tau_i$ are fractional ideals in K representing the different ideal classes of K. Kronecker proved the field $K(j(\tau_1))$ is a Galois extension of K whose Galois group is isomorphic to the ideal class group of K. How can the ideal class group of K be identified with the Galois group of $K(j(\tau_1))/K$? Let a fractional ideal $\mathfrak{b}$ act on $j(\tau_i)$ using multiplication in the class group: if $\mathfrak{b}(\mathbf{Z} + \mathbf{Z}\tau_i) = \mathbf{Z} + \mathbf{Z}\tau_{i'}$ in $\mathrm{Cl}(K)$ then set $\sigma_{\mathfrak{b}}(j(\tau_i)) = j(\tau_{i'})$. This action of fractional ideals on the j-values descends to an action of the ideal class group on the j-values.

Example 2.2. Let $K = \mathbf{Q}(\sqrt{-31})$. The class number is 3 and ideals representing the different ideal classes are (1), $\mathfrak{p}_2$, $\overline{\mathfrak{p}}_2$, where $\mathfrak{p}_2 = 2\mathbf{Z} + (\frac{1+\sqrt{-31}}{2})\mathbf{Z}$ and $\overline{\mathfrak{p}}_2$ is the conjugate ideal of $\mathfrak{p}_2$. Scaling each ideal to be a lattice of the form $\mathbf{Z}+\mathbf{Z}\tau$ for τ in the upper half-plane, three values of τ are $\frac{1+\sqrt{-31}}{2}$, $\frac{1+\sqrt{-31}}{4}$, and $\frac{-1+\sqrt{-31}}{4}$. The values of the j-function at these three numbers are the roots of the cubic polynomial

$$X^3 + 39491307X^2 - 58682638134X + 1566028350940383.$$

Its discriminant is $-31\cdot(3^{19}\cdot 11^2\cdot 13^3\cdot 17\cdot 23\cdot 29)^2$, so its splitting field over $\mathbf{Q}$ is $\mathbf{Q}(\gamma, \sqrt{-31})$, where $\gamma = j(\frac{1+\sqrt{-31}}{2}) \approx -39492793.9115$. The extension $K(\gamma)/K$ is a cubic Galois extension generated by a special value of the j-function.

Kronecker called $K(j(\tau))$, where $\mathcal{O}_K = \mathbf{Z}+\mathbf{Z}\tau$, the "species" associated to K, continuing the co-opting of taxonomic terminology in number theory (earlier examples being class, order, and genus). In examples, Kronecker observed the species of K is not just an extension of K with Galois group isomorphic to the ideal class group, but has two other properties: it is unramified over K and every ideal of K becomes principal in it. Hilbert will include these properties as part of his general conjectures on Hilbert class fields.

[2]Kronecker was 56 at the time, and was in his 30s when he worked on the Kronecker–Weber theorem and relations between modular functions and imaginary quadratic fields.

In addition to the construction of abelian extensions, Kronecker set off another path to class field theory in an 1880 paper on densities of primes and factorization of polynomials. For a polynomial $f(X) \in \mathbf{Z}[X]$, Kronecker considered the number n_p of roots of $f(X)$ mod p in $\mathbf{F}_p$ as p varies. For example, if $f(X) = X^2 + 1$ then $n_2 = 1$ and $n_p = 0$ or 2 for odd p, depending on p modulo 4. So on average, n_p is 1.

Theorem 2.3 (Kronecker, 1880)**.** *If $f(X)$ has r irreducible factors in $\mathbf{Z}[X]$ then the average value of n_p is r:*

$$\lim_{s \to 1^+} \frac{\sum_p n_p/p^s}{\sum_p 1/p^s} = r.$$

The use of a density with Dirichlet series rather than as $\lim_{x\to\infty}(\sum_{p\leq x} n_p)/|\{p \leq x\}|$ is no surprise: the Prime Number theorem was still 16 years in the future, so a rigorous notion of density at the time could not use denominator $|\{p \leq x\}|$.

Corollary 2.4. *Let $K/\mathbf{Q}$ be a Galois extension. The set of primes that split completely in K has density $1/[K : \mathbf{Q}]$.*

Proof. Write $K = \mathbf{Q}(\alpha)$ for an algebraic integer α. Let $f(X) \in \mathbf{Z}[X]$ be the minimal polynomial of α over $\mathbf{Q}$. Because the roots of $f(X)$ are polynomials in α with rational coefficients, if $f(X)$ mod p has a root then it splits completely (for all but finitely many p), which means $n_p = \deg f = [K : \mathbf{Q}]$ if $n_p \neq 0$. Letting A be the set of primes p such $n_p = [K : \mathbf{Q}]$, Kronecker's theorem says in this case

$$\lim_{s \to 1^+} \frac{\sum_{p\in A} 1/p^s}{\sum_p 1/p^s} = \frac{1}{[K : \mathbf{Q}]},$$

so the (Dirichlet) density of the p where $f(X)$ mod p splits completely is $1/[K : \mathbf{Q}]$. Since $f(X)$ mod p splits completely if and only if p splits completely in K (with finitely many exceptions), the primes that split completely in K have density $1/[K : \mathbf{Q}]$. □

Example 2.5. Kronecker used Corollary 2.4 to prove irreducibility of the nth cyclotomic polynomial by the following analytic method. Let $K = \mathbf{Q}(\zeta_n)$, which is Galois over $\mathbf{Q}$ since all nth roots of unity are powers of ζ_n. Let $f(X)$ be the minimal polynomial of ζ_n in $\mathbf{Z}[X]$. By Corollary 2.4, the set of primes that split completely in K has density $1/[K : \mathbf{Q}] = 1/\deg f(X)$. What are these primes? For all but finitely many p, p splits completely in K if and only if $f(X)$ splits completely in $\mathbf{F}_p[X]$, which is equivalent (for all p not dividing n) to there being a primitive nth root of unity in $\mathbf{F}_p$. That means $n \mid (p-1)$, or $p \equiv 1 \bmod n$. So a prime p splits completely in K if and only if $p \equiv 1 \bmod n$, with perhaps a finite number of exceptions. By Dirichlet's theorem, the set of primes p satisfying $p \equiv 1 \bmod n$ has density $1/\varphi(n)$. Therefore $\deg f(X) = \varphi(n)$. Since ζ_n is a root of the nth cyclotomic polynomial $\Phi_n(X)$, which has degree $\varphi(n)$ and is monic in $\mathbf{Z}[X]$, this proves $f(X) = \Phi_n(X)$, so $\Phi_n(X)$ is irreducible in $\mathbf{Q}[X]$.

Kronecker's paper included two influential conjectures on sets of primes. The first asked, for $f(X) \in \mathbf{Z}[X]$, what the density is of the set of primes p such that $f(X)$ mod p has a fixed number of roots in $\mathbf{F}_p$. When that number is $\deg f$, so $f(X)$ mod p splits completely, Kronecker found the density. He could not prove the densities exist in general, but he conjectured they do and what some of their properties are. The existence of these densities was proved by Frobenius, and in his work on this problem Frobenius introduced (1896) the

Frobenius element of a prime ideal[3] and conjectured what became the Chebotarev density theorem. We will see its importance when we discuss Artin's work on class field theory.

Kronecker's second conjecture was that a Galois extension of $\mathbf{Q}$ is characterized by the set of primes in $\mathbf{Q}$ that split completely in the extension (*e.g.*, $\mathbf{Q}(i)$ is the only Galois extension of $\mathbf{Q}$ in which the split primes are $p \equiv 1 \bmod 4$). He considered this idea as an arithmetic "boundary value theorem," just as Cauchy's integral formula determines a complex analytic function inside a disc from its values on the boundary. For a finite extension L/K, denote the set of primes in K that split in L as $\mathrm{Spl}(L/K)$. Corollary 2.4 generalizes to the following result: when L/K is Galois, $\mathrm{Spl}(L/K)$ has Dirichlet density $1/[L:K]$, where the Dirichlet density of a set A of prime ideals in K is

$$\lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in A} 1/(\mathrm{N}\mathfrak{p})^s}{\sum_{\mathfrak{p}} 1/(\mathrm{N}\mathfrak{p})^s}$$

if that limit exists. The set of prime ideals in K with non-prime norm has Dirichlet density 0, so if A has a Dirichlet density then the set of prime ideals in A with prime norm has the same Dirichlet density. Kronecker's second conjecture was proved by M. Bauer in 1903 for Galois extensions of all number fields, not just $\mathbf{Q}$.

Theorem 2.6 (Bauer)**.** *Let L_1 and L_2 be finite Galois extensions of a number field K. Then $L_1 \subset L_2$ if and only if $\mathrm{Spl}(L_2/K) \subset \mathrm{Spl}(L_1/K)$. In particular, $L_1 = L_2$ if and only if $\mathrm{Spl}(L_1/K) = \mathrm{Spl}(L_2/K)$.*

Proof. If $L_1 \subset L_2$ then easily $\mathrm{Spl}(L_2/K) \subset \mathrm{Spl}(L_1/K)$. Conversely, if $\mathrm{Spl}(L_2/K) \subset \mathrm{Spl}(L_1/K)$, consider the extension L_1L_2/K. It is Galois, and $\mathrm{Spl}(L_1L_2/K) = \mathrm{Spl}(L_1/K) \cap \mathrm{Spl}(L_2/K)$. From the hypothesis, $\mathrm{Spl}(L_1/K) \cap \mathrm{Spl}(L_2/K) = \mathrm{Spl}(L_2/K)$, so $\mathrm{Spl}(L_1L_2/K) = \mathrm{Spl}(L_2/K)$. Computing the Dirichlet density of both sides, we get $1/[L_1L_2 : K] = 1/[L_2 : K]$ by Corollary 2.4. Thus $L_1L_2 = L_2$, so $L_1 \subset L_2$. □

Changing a set of primes by a finite amount does not affect its density, so Bauer's theorem is true with the inclusion and equality of sets of split primes being true up to finitely many exceptions. For example, the only Galois extension of $\mathbf{Q}$ whose split primes are $\{p \equiv 1 \bmod 4\}$ up to finitely many exceptional primes is $\mathbf{Q}(i)$. Allowing finitely many exceptional primes in Bauer's theorem is important when we use it in the context of Weber's definition of a class field below.

Although Bauer's theorem tells us that a Galois extension L/K is determined (as an extension of K) by the primes in K that split completely in L, this doesn't give us a simple rule for describing the set of split primes. When L/K is abelian, class field theory will give a simple rule for $\mathrm{Spl}(L/K)$ in terms of generalized congruences.

3. Splitting Laws (Weber)

Reference: [2].

In his 1891 book on elliptic functions and algebraic numbers, H. Weber introduced the label "class field" for Kronecker's species, so at first a class field meant a particular abelian (conjecturally unramified) extension of an imaginary quadratic field, whose Galois group is isomorphic to the ideal class group of the base field. In 1897, Weber extended the notion of

[3]The construction of Frobenius elements was made by Frobenius in 1880 and independently found by Dedekind, according to correspondence between them in 1882.

ideal class group: for a number field K and nonzero ideal $\mathfrak{m}$ in $\mathcal{O}_K$, let $I_\mathfrak{m}$ be the group of fractional ideals in K relatively prime to $\mathfrak{m}$ and let $P_\mathfrak{m}^+$ be the group of principal fractional ideals having some generator α/β for nonzero α and β in $\mathcal{O}_K$ such that

- (α) and (β) are relatively prime to $\mathfrak{m}$,
- $\alpha \equiv \beta \bmod \mathfrak{m}$.
- α/β is totally positive (that is, under all real embeddings of K, if there are any, α/β has positive image).

Example 3.1. An ideal is in $P_{(1)}^+$ if it has a totally positive generator. In $\mathbf{Q}(\sqrt{2})$, the ideal $(\sqrt{2})$ is in $P_{(1)}^+$ even though $\sqrt{2}$ is not totally positive, since another generator $\sqrt{2}(1+\sqrt{2})$ is totally positive. In $\mathbf{Q}(\sqrt{3})$, $(\sqrt{3}) \notin P_{(1)}^+$ since $\sqrt{3}u$ is not totally positive for all $u \in \mathbf{Z}[\sqrt{3}]^\times$: the units in $\mathbf{Z}[\sqrt{3}]$ have norm 1, so $\sqrt{3}u$ has norm -3 and therefore can't be totally positive.

The index $[I_\mathfrak{m} : P_\mathfrak{m}^+]$ is finite. An intermediate group H, where

$$P_\mathfrak{m}^+ \subset H \subset I_\mathfrak{m},$$

is called an *ideal group with modulus* $\mathfrak{m}$ and $I_\mathfrak{m}/H$ is called a *generalized ideal class group.*

Example 3.2. If $\mathfrak{m} = (1)$ and P is the group of principal fractional ideals, then $P_{(1)}^+ \subset P \subset I_{(1)}$ and $I_{(1)}/P$ is the ideal class group. The quotient $I_{(1)}/P_{(1)}^+$ is called the *narrow ideal class group*: it identifies two fractional ideals equal up to multiplication not by a principal ideal with a *totally positive* generator. For example, $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ have trivial ideal class groups, $\mathbf{Q}(\sqrt{2})$ has a trivial narrow ideal class group (all nonzero ideals in $\mathbf{Z}[\sqrt{2}]$ are principal with a totally positive generator), and $\mathbf{Q}(\sqrt{3})$ has a narrow ideal class group of order 2 represented by (1) and $(\sqrt{3})$: see Example 3.1. The ideal class group is called the *wide ideal class group* to contrast it with the narrow ideal class group.

Example 3.3. When $K = \mathbf{Q}$ and $\mathfrak{m} = m\mathbf{Z}$, $I_\mathfrak{m}/P_\mathfrak{m}^+ \cong (\mathbf{Z}/m\mathbf{Z})^\times$: an element of $(\mathbf{Z}/m\mathbf{Z})^\times$ is $a \bmod m\mathbf{Z}$ where $a > 0$ and $(a, m) = 1$. Check the map $f\colon (\mathbf{Z}/m\mathbf{Z})^\times \to I_\mathfrak{m}/P_\mathfrak{m}^+$ where $f(a \bmod m\mathbf{Z}) = (a\mathbf{Z})P_\mathfrak{m}^+$ and we require $a > 0$ is (i) well-defined, (ii) a group homomorphism, and (iii) bijective. (It is onto since an ideal in $I_\mathfrak{m}$ is $(b/c)\mathbf{Z}$ where $(b, m) = 1$ and $(c, m) = 1$ with $f(\pm b \bmod m\mathbf{Z}) = (b\mathbf{Z})P_\mathfrak{m}^+$ and $f(\pm c \bmod m\mathbf{Z}) = (c\mathbf{Z})P_\mathfrak{m}^+$ for some signs. To see f is injective, if $f(a \bmod m\mathbf{Z}) = P_\mathfrak{m}^+$ and $a > 0$ then $a\mathbf{Z} \in P_\mathfrak{m}^+$, so $a \equiv 1 \bmod m$.)

Equivalence classes of quadratic forms, under the operation of Gauss composition, are isomorphic to certain generalized ideal class groups of quadratic fields.

Since $I_\mathfrak{m}/H$ generalizes $(\mathbf{Z}/m\mathbf{Z})^\times$, Weber sought an analogue of Dirichlet's theorem (1837) that each congruence class in $(\mathbf{Z}/m\mathbf{Z})^\times$ has infinitely many primes: each coset in $I_\mathfrak{m}/H$ has infinitely many prime ideals. To prove that, Weber adapted Dirichlet's method based on L-functions of characters $\chi\colon (\mathbf{Z}/m\mathbf{Z})^\times \to \mathbf{C}^\times$, where

$$L(s, \chi) := \prod_{(p,m)=1} \frac{1}{1 - \chi(p)p^{-s}} = \sum_{(n,m)=1} \frac{\chi(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$. The series converges (conditionally) for $\mathrm{Re}(s) > 0$ when χ is nontrivial, and the heart of Dirichlet's argument is the proof that $L(1, \chi) \neq 0$ for all nontrivial χ. Weber introduced an L-function for characters $\psi\colon I_\mathfrak{m}/H \to \mathbf{C}^\times$:

$$L(s, \psi) = \prod_{\mathfrak{p} \nmid \mathfrak{m}} \frac{1}{1 - \psi(\mathfrak{p})\,\mathrm{N}\mathfrak{p}^{-s}} = \sum_{(\mathfrak{a},\mathfrak{m})=1} \frac{\psi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s} \tag{3.1}$$

for $\mathrm{Re}(s) > 1$. When ψ is trivial, this is essentially the Dedekind zeta-function of K. By studying the behavior of $L(s, \psi)$ as $s \to 1^+$ for all characters ψ of $I_{\mathfrak{m}}/H$, Weber proved the following result.

Theorem 3.4 (Weber)**.** *For a nonzero ideal $\mathfrak{m}$ in $\mathcal{O}_K$ and ideal group H with modulus $\mathfrak{m}$, assume there is a Galois extension L/K such that $\mathrm{Spl}(L/K) \subset H$ with finitely many exceptions. Then*

$$[I_{\mathfrak{m}} : H] \leq [L : K]. \tag{3.2}$$

If $\mathrm{Spl}(L/K) = H$ with finitely many exceptions then $[I_{\mathfrak{m}} : H] = [L : K]$ and the set of primes in each coset of $I_{\mathfrak{m}}/H$ has Dirichlet density $1/[I_{\mathfrak{m}} : H]$. In particular, there are infinitely many primes in each coset of $I_{\mathfrak{m}}/H$.

Definition 3.5 (Weber, 1908)**.** For a nonzero ideal $\mathfrak{m}$ in $\mathcal{O}_K$ and ideal group H with modulus $\mathfrak{m}$, the *class field* over K for H is a Galois extension L/K such that for primes $\mathfrak{p} \nmid \mathfrak{m}$ in K,

$$\mathfrak{p} \text{ splits completely in } L \Longleftrightarrow \mathfrak{p} \in H. \tag{3.3}$$

A class field over K, for Weber, is an extension of K whose set of split primes are (up to finitely many exceptions) the prime ideals in an ideal group. We have to allow finitely many exceptions because an ideal group with modulus $\mathfrak{m}$ contains no prime dividing $\mathfrak{m}$ (they are not in $I_{\mathfrak{m}}$) and some of the prime factors of $\mathfrak{m}$ may split in the extension.

Theorem 3.4 tells us what class fields can be good for: their existence implies the infinitude of primes in cosets of generalized ideal class groups. Bauer's theorem (allowing finitely many exceptional primes) tells us the class field L/K for H is unique *if* it exists.

Example 3.6. The extension $\mathbf{Q}(i)/\mathbf{Q}$ is the class field for $P_{(4)}^+$ since p splits in $\mathbf{Q}(i)$ if and only if $p \equiv 1 \bmod 4$ (where p is a positive prime).

Example 3.7. The extension $\mathbf{Q}(\sqrt{2})/\mathbf{Q}$ is the class field for $\{P_8^+, -P_{(8)}^+\}$, since p splits in $\mathbf{Q}(\sqrt{2})$ if and only if $p \equiv \pm 1 \bmod 8$.

Weber needed the existence of class fields to prove his extension of Dirichlet's theorem to generalized ideal class groups. For certain ideal groups H from imaginary quadratic K, Weber could prove the existence of a class field L/K and an isomorphism of $\mathrm{Gal}(L/K)$ with $I_{\mathfrak{m}}/H$, but the existence of class fields for ideal groups over number fields beyond $\mathbf{Q}$ and imaginary quadratics was left open.

Example 3.8. Let K be an imaginary quadratic field and $\mathfrak{m}$ be a nonzero ideal in $\mathcal{O}_K$. Since K has no real embeddings, the third condition in the definition of $P_{\mathfrak{m}}^+$ is empty: this is the group of principal fractional ideals with a generator α/β where α and β are nonzero in $\mathcal{O}_K$, (α) and (β) are relatively prime to $\mathfrak{m}$, and $\alpha \equiv \beta \bmod \mathfrak{m}$. Check an ideal belongs to $P_{\mathfrak{m}}^+$ if and only if it is principal with a generator γ such that $\gamma \equiv 1 \bmod \mathfrak{m}$. If $P_{\mathfrak{m}}^+$ has a class field L, then Theorem 3.4 tells us that the set of prime ideals in $P_{\mathfrak{m}}^+$ has Dirichlet density $1/[L : K]$. That says the set of principal prime ideals (γ) such that $\gamma \equiv 1 \bmod \mathfrak{m}$ has Dirichlet density $1/[L : K]$. The set of prime ideals in K with non-prime norm has Dirichlet density 0, and $\mathrm{N}((\gamma)) = |\mathrm{N}_{K/\mathbf{Q}}(\gamma)| = \mathrm{N}_{K/\mathbf{Q}}(\gamma)$ since $K/\mathbf{Q}$ is imaginary quadratic, so the set of principal prime ideals (γ) in $\mathcal{O}_K$ such that (i) $\gamma \equiv 1 \bmod \mathfrak{m}$ and (ii) $\mathrm{N}(\gamma)$ is a prime number has Dirichlet density $1/[L : K]$.

4. Unramified Extensions (Hilbert)

References: [13], [19], [22].

Hilbert's ideas about abelian extensions of number fields developed from his careful study of three families of examples: quadratic and cyclotomic extensions of general number fields and Kummer extensions of cyclotomic fields. One of his goals was to develop reciprocity laws in number fields, building on his conception (1897) of the quadratic reciprocity law over $\mathbf{Q}$ as a product formula: $\prod_v (a,b)_v = 1$ for all a and b in $\mathbf{Q}^\times$, where $(a,b)_v$ is the v-adic Hilbert symbol:

$$(a,b)_v = \begin{cases} 1, & \text{if } a = x^2 - by^2 \text{ is solvable in } \mathbf{Q}_v, \\ -1, & \text{otherwise.} \end{cases}$$

This is equivalent to quadratic reciprocity, but nicer in two respects: the prime 2 is on the same footing as the other primes and there are no positivity or relative primality constraints on a and b. There are several new aspects being used in this version of quadratic reciprocity: emphasis on norms rather squares, p-adic equations[4] rather than congruences modulo p, and the infinite places on an equal footing with the finite places. Ultimately all three ideas will appear in class field theory, and none are present in Weber's work.

The Hilbert symbol makes sense on all number fields K (replacing completions of $\mathbf{Q}$ with completions of K in the definition), so Hilbert proposed a quadratic reciprocity law on K: $\prod_v (a,b)_v = 1$ for a and b in $K^\times$ with v running over all the places of K. Hilbert's proof of this formula broke down for number fields that admit a quadratic extension unramified at all primes. This doesn't mean the result is wrong for those fields, only that the proof doesn't work. An obstruction like this had happened before: Kummer's p-th power reciprocity law in $\mathbf{Q}(\zeta_p)$ (1859) was restricted to regular primes ($p \nmid h(\mathbf{Q}(\zeta_p))$), which Hilbert could interpret as avoiding the cases when $\mathbf{Q}(\zeta_p)$ has an abelian unramified extension of degree p. Hilbert's proof of the Kronecker-Weber theorem succeeded in part because $\mathbf{Q}$ has no (abelian) unramified extensions larger than $\mathbf{Q}$. It is perhaps this experience that drove Hilbert's interest in *unramified* abelian extensions, as an obstacle in proofs. Thinking about analogies between number fields and Riemann surfaces (*e.g.*, prime ideals correspond to points and unramified extensions of number fields correspond to unbranched coverings of Riemann surfaces), Hilbert was led to the following conjecture.

Conjecture 4.1 (Hilbert, 1898)**.** *For each number field K there is a unique finite extension K'/K such that*

1. *K'/K is Galois and $\mathrm{Gal}(K'/K) \cong \mathrm{Cl}(K)$,*
2. *K'/K is unramified at all places, and every abelian extension of K with this property is a subfield of K',*
3. *for each prime $\mathfrak{p}$ of K, the residue field degree $f_{\mathfrak{p}}(K'/K)$ is the order of $\mathfrak{p}$ in $\mathrm{Cl}(K)$,*
4. *every ideal of K is principal in K'.*

Condition (3) implies that a prime in K splits in K' if and only if it is principal in K, so K' is a class field over K in Weber's sense for the ideal group of all principal fractional

[4]Hilbert wrote congruences modulo arbitrarily high powers of p, as the p-adics did not yet exist, although they were just on the verge of existence: Hensel's first paper on them appeared in 1897.

ideals in K. The field K' is called the *Hilbert class field* of K, but Hilbert just called it a "class field." Kronecker's species of an imaginary quadratic field is its Hilbert class field.[5]

Example 4.2. The extensions

$$\begin{array}{cccc} \mathbf{Q}(\sqrt{10},\sqrt{2}) & \mathbf{Q}(\sqrt{-23},\alpha) & \mathbf{Q}(\sqrt{-31},\beta) & \mathbf{Q}(\sqrt{-14},\sqrt{2\sqrt{2}-1}) \\ \Big| 2 & \Big| 3 & \Big| 3 & \Big| 4 \\ \mathbf{Q}(\sqrt{10}) & \mathbf{Q}(\sqrt{-23}) & \mathbf{Q}(\sqrt{-31}) & \mathbf{Q}(\sqrt{-14}) \end{array}$$

are all examples of Hilbert class fields over the base fields, with $\alpha^3 - \alpha - 1 = 0$ and $\beta^3 + \beta + 1 = 0$. In particular, each of these extensions has degree equal to the class number of the base field and the Galois group of the extension is isomorphic to the ideal class group of the base field. The third example is the same as $\mathbf{Q}(\sqrt{-31},\gamma)$ in Example 2.2, since $\beta := -16480503\gamma^2 + 11239722\gamma - 24150771$ is a root of $X^3 + X + 1$.

Hilbert proved Conjecture 4.1 whenever $h(K) = 2$ and he formulated Conjecture 4.1 as a natural generalization of that work. In 1907, Furtwängler proved the first two parts of Conjecture 4.1 in general and later used this to prove a p-th power reciprocity law in all number fields (which for $p = 2$ is a quadratic reciprocity law). He proved the third part in 1911. The fourth part, which extends an observation of Kronecker for imaginary quadratic fields, was proved by Furtwängler (1930) after Artin reduced it to a purely group-theoretic statement related to the iterated Hilbert class field $K'' = (K')'$, which is a Galois extension of K that is usually not abelian.

5. Class Field Theory Proved (Takagi)

References: [6], [8], [16]

T. Takagi studied in Germany during 1898–1901, partly with Hilbert in Göttingen. In his 1903 thesis, Takagi proved the Jugendtraum for base field $\mathbf{Q}(i)$ using values of the lemniscatic function, as Kronecker had envisioned. His proof was an adaptation to $\mathbf{Q}(i)$ of Hilbert's proof of the Kronecker–Weber theorem. In 1914, R. Fueter proved that for each imaginary quadratic field K, viewed as a subfield of $\mathbf{C}$, every *odd*-degree abelian extension of K inside of $\mathbf{C}$ is a subfield of some $K(e^{2\pi ir}, j(\tau))$, where $r \in \mathbf{Q}$ and $\tau \in K$ (with $\mathrm{Im}(\tau) > 0$). In other words, all odd degree abelian extensions of K are inside fields generated over K by special values of two analytic functions at algebraic numbers: the exponential function $e^{2\pi iz}$ at rational numbers and the j-function at numbers in K. Fueter also gave a counterexample for extensions of even degree: $\mathbf{Q}(\sqrt[4]{1+2i})$ has degree 4 over $\mathbf{Q}(i)$ and is a cyclic extension, but it lies in no field of the form $\mathbf{Q}(i, e^{2\pi ir}, j(\tau))$ for $r \in \mathbf{Q}$ and $\tau \in \mathbf{Q}(i)$.

Takagi read the work of Furtwängler on the Hilbert class field and Fueter on the Jugendtraum over imaginary quadratic fields. When World War I broke out in 1914, scientific contact between Germany (the only place where algebraic number theory was under serious

[5]Hilbert's notion of class field was an abelian extension unramified at all prime ideals, allowing ramification at infinity. For instance, $\mathbf{Q}(\sqrt{3})$ and $\mathbf{Q}(\sqrt{6})$ have class number 1 but $\mathbf{Q}(\sqrt{3},i)/\mathbf{Q}(\sqrt{3})$ and $\mathbf{Q}(\sqrt{6},\sqrt{-2})/\mathbf{Q}(\sqrt{6})$ are unramified at all prime ideals in $\mathbf{Z}[\sqrt{3}]$ and $\mathbf{Z}[\sqrt{6}]$; there is ramification at infinity. Conjecture 4.1 by Hilbert was about the "Hilbert class field in the narrow sense," whose Galois group is isomorphic to the narrow ideal class group where fractional ideals are identified only if their ratio is a principal ideal having a *totally positive* generator. The narrow class numbers of $\mathbf{Q}(\sqrt{3})$ and $\mathbf{Q}(\sqrt{6})$ are 2. Our version of Conjecture 4.1 includes unramifiedness at the infinite places since that's what the Hilbert class field means today.

study) and Japan ceased. Working in isolation, Takagi combined the work of Furtwängler and Fueter with an inductive procedure to prove the existence of class fields in full generality, and nearly everything else that was expected about them.

Takagi began with a new definition of a class field, using norms of ideals, not splitting laws, and using infinite places in the modulus. (Note the influence of Hilbert's ideas.)

Definition 5.1. For a finite extension of number fields L/K and a prime $\mathfrak{P}$ of L, let $\mathfrak{p} = \mathfrak{P} \cap K$ be the prime below it in K and set the *norm* of $\mathfrak{P}$ in K to be

$$\mathrm{N}_{L/K}(\mathfrak{P}) := \mathfrak{p}^{f(\mathfrak{P}|\mathfrak{p})}.$$

Extend the norm by multiplicativity to all fractional ideals in L.

Such norms play a key role in class field theory. On principal fractional ideals they are compatible with the ring-theoretic norm: $\mathrm{N}_{L/K}(x\mathcal{O}_L) = \mathrm{N}_{L/K}(x)\mathcal{O}_K$ for $x \in L^\times$.

The link between Weber's and Takagi's viewpoints is that for Galois L/K and $\mathfrak{p}$ in K unramified in L, $\mathfrak{p}$ splits in L (Weber) if and only if $\mathfrak{p}$ is the norm of a prime in $\mathcal{O}_L$ (Takagi).

Definition 5.2. A *K-modulus* is a formal product $\mathfrak{m} = \mathfrak{m}_f\mathfrak{m}_\infty$, where $\mathfrak{m}_f$ (the "finite part") is a nonzero ideal in $\mathcal{O}_K$ and $\mathfrak{m}_\infty$ is a formal product of real embeddings of K. A fractional ideal of K is called *relatively prime* to $\mathfrak{m}$ when it is relatively prime to $\mathfrak{m}_f$.

Let $I_\mathfrak{m}$ be the fractional ideals relatively prime to $\mathfrak{m}$ and $P_\mathfrak{m}$ be the principal fractional ideals having some generator α/β for nonzero α and β in $\mathcal{O}_K$ such that[6]

- (α) and (β) are relatively prime to $\mathfrak{m}$ (meaning they're relatively prime to $\mathfrak{m}_f$),
- $\alpha \equiv \beta \bmod \mathfrak{m}_f$,
- $v(\alpha/\beta) > 0$ for all real embeddings $v \mid \mathfrak{m}_\infty$.[7]

Example 5.3. When $K = \mathbf{Q}$, a K-modulus is either $(m) := m\mathbf{Z}$ or $(m\infty) := (m\mathbf{Z})\infty$ where $m \geq 1$. We have $I_{(m\infty)} = I_{(m)}$ and $P_{(m\infty)} = P^+_{(m)}$, but $P_{(m\infty)} \neq P_{(m)}$ for $m \geq 3$: $(m-1)\mathbf{Z} = (1-m)\mathbf{Z} \in P_{(m)}$ and $(m-1)\mathbf{Z} \notin P_{(m\infty)}$ since the only positive generator of $(m-1)\mathbf{Z}$ is not 1 mod m. We have $P_{(m\infty)} = P_{(m)}$ when m is 1 or 2.

For $m \geq 1$, $I_{(m\infty)}/P_{(m\infty)} = I_{(m)}/P^+_{(m)} \cong (\mathbf{Z}/m\mathbf{Z})^\times$ by Example 3.3. What is $I_{(m)}/P_{(m)}$, concretely? The map $f\colon (\mathbf{Z}/m\mathbf{Z})^\times \to I_{(m)}/P_{(m)}$ where $f(a \bmod m) = (a\mathbf{Z})P_{(m)}$ with no sign condition on a is (i) well-defined, (ii) a group homomorphism, and (iii) surjective by reasoning as in Example 3.3. If $a \bmod m \in \ker f$ then $a\mathbf{Z} \in P_{(m)}$, so $\pm a \equiv \pm 1 \bmod m$. Thus $\ker f = \{\pm 1 \bmod m\}$, so $I_{(m)}/P_{(m)} \cong (\mathbf{Z}/m\mathbf{Z})^\times/\{\pm 1\}$ for $m \geq 1$.

As an exercise, check that when K has class number 1 and (α) is a nonzero ideal in $\mathcal{O}_K$, $I_{(\alpha)}/P_{(\alpha)} \cong (\mathcal{O}_K/(\alpha))^\times/U$ as groups where U is the image of the natural reduction map $\mathcal{O}_K^\times \to (\mathcal{O}_K/(\alpha))^\times$. Here (α) regarded as a K-modulus has no real embeddings.

For a K-modulus $\mathfrak{m}$, an intermediate group H where $P_\mathfrak{m} \subset H \subset I_\mathfrak{m}$ is called an *ideal group with modulus* $\mathfrak{m}$. For a finite extension L/K, set

$$\mathrm{N}_\mathfrak{m}(L/K) := \{\mathfrak{a} \text{ in } K : \mathfrak{a} = \mathrm{N}_{L/K}(\mathfrak{A}) \text{ for a fractional ideal } \mathfrak{A} \text{ in } L, \mathfrak{a} \text{ is rel. prime to } \mathfrak{m}\}.$$

and

$$H_\mathfrak{m}(L/K) := P_\mathfrak{m}\,\mathrm{N}_\mathfrak{m}(L/K).$$

[6] A description of $P_\mathfrak{m}$ using just α/β, not α and β separately, is in Corollary A.5.

[7] For an ideal $\mathfrak{n} \neq (0)$ in $\mathcal{O}_K$, $P^+_\mathfrak{n}$ in Section 3 is $P_{\mathfrak{n}\infty}$ here, where ∞ is the product of real embeddings of K.

What is the purpose of the group $H_{\mathfrak{m}}(L/K)$? We want to create an ideal group using norms of ideals, but $\mathrm{N}_{\mathfrak{m}}(L/K)$ need not contain $P_{\mathfrak{m}}$. Multiplying by $P_{\mathfrak{m}}$ gives us a group that contains $P_{\mathfrak{m}}$ and thus becomes an ideal group. Put differently, the subgroup of $I_{\mathfrak{m}}/P_{\mathfrak{m}}$ generated by cosets of $\mathrm{N}_{\mathfrak{m}}(L/K)$ is $H_{\mathfrak{m}}(L/K)/P_{\mathfrak{m}}$, so this quotient group is the "norm subgroup" of $I_{\mathfrak{m}}/P_{\mathfrak{m}}$ that Takagi is focusing on. It eventually turns out that every subgroup of $I_{\mathfrak{m}}/P_{\mathfrak{m}}$ is such a norm subgroup for some finite abelian extension of K.

How are Weber's and Takagi's ideal groups related? For a number field K and nonzero ideal $\mathfrak{m}$ in $\mathcal{O}_K$, Weber's $P_{\mathfrak{m}}^+$ equals Takagi's $P_{\mathfrak{m}\infty}$, where ∞ is the product of all real places of the number field. Takagi has a more general construction than Weber since Takagi allows varying sign conditions in his modulus. (Weber mainly worked over imaginary quadratic fields, which have no real places, there was no motivation to be sensitive to sign conditions.)

When $\mathfrak{m}$ is a K-modulus and L/K is Galois, the primes of K not dividing $\mathfrak{m}$ that split in L lie in $\mathrm{N}_{\mathfrak{m}}(L/K) \subset H_{\mathfrak{m}}(L/K)$, so $\mathrm{Spl}(L/K) \subset H_{\mathfrak{m}}(L/K)$ except perhaps for primes dividing $\mathfrak{m}$. Therefore Theorem 3.4 (whose proof works with a K-modulus in place of the nonzero ideals in $\mathcal{O}_K$) implies

$$[I_{\mathfrak{m}} : H_{\mathfrak{m}}(L/K)] \leq [L : K]. \tag{5.1}$$

Definition 5.4 (Takagi)**.** A Galois extension of number fields L/K is called a *class field* when (5.1) has equality for a K-modulus $\mathfrak{m}$. Call such $\mathfrak{m}$ an *admissible* modulus for L/K.

Example 5.5. We saw in Example 3.6 that $\mathbf{Q}(i)/\mathbf{Q}$ is a class field in Weber's sense. Now we will show it is a class field in Takagi's sense, with admissible modulus 4∞. Since $\mathrm{N}_{4\infty}(\mathbf{Q}(i)/\mathbf{Q}) \subset P_{4\infty}$ (essentially because an odd sum of two integral squares is 1 mod 4), $H_{4\infty}(\mathbf{Q}(i)/\mathbf{Q}) = P_{4\infty}$, and $[I_{4\infty} : P_{4\infty}] = |(\mathbf{Z}/4\mathbf{Z})^\times| = 2 = [\mathbf{Q}(i) : \mathbf{Q}]$, so the upper bound is reached.

To define a class field, Weber picks an ideal group H and seeks a corresponding (class) field L/K, which should exist and be abelian, while Takagi picks an L/K and sees if there is an ideal group H making (5.1) an equality (which may not happen).

It is useful to know the relations among all admissible moduli for an extension L/K. Each multiple of an admissible modulus is admissible.[8] To prove this, if $\mathfrak{m} \mid \mathfrak{m}'$ then $I_{\mathfrak{m}'} \subset I_{\mathfrak{m}}$, $P_{\mathfrak{m}'} \subset P_{\mathfrak{m}}$, and $\mathrm{N}_{\mathfrak{m}'}(L/K) \subset \mathrm{N}_{\mathfrak{m}}(L/K)$, so $H_{\mathfrak{m}'}(L/K) \subset H_{\mathfrak{m}}(L/K)$. The natural map $I_{\mathfrak{m}'}/H_{\mathfrak{m}'} \to I_{\mathfrak{m}}/H_{\mathfrak{m}}$ is onto,[9] so (5.1) with modulus $\mathfrak{m}'$ implies $[I_{\mathfrak{m}'} : H_{\mathfrak{m}'}] = [L : K]$ if $[I_{\mathfrak{m}} : H_{\mathfrak{m}}] = [L : K]$. For two admissible K-moduli, their least common multiple is admissible. Going the other way, the greatest common factor of two admissible moduli is admissible (this is somewhat more delicate to show, since not every factor of an admissible modulus is admissible), so we can speak about the least admissible modulus: there is a K-modulus that is admissible for L/K and the admissible moduli for L/K are precisely the multiples of it. The least admissible modulus for L/K is called the *conductor* of L/K and is denoted $\mathfrak{f}_{L/K}$ from the German word Führer.[10]

[8]Using Weber's definition of a class field for an ideal group, call an ideal $\mathfrak{m}$ in $\mathcal{O}_K$ *admissible* for L/K if there is an ideal group with modulus $\mathfrak{m}$ in Weber's sense whose class field in Weber's sense is L. The notions of admissible for Weber and Takagi ultimately turn out to coincide, but it's a good exercise to check directly from the definition that each multiple of an admissible modulus in Weber's sense is admissible.

[9]This means each coset in $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ contains an ideal relatively prime to $\mathfrak{m}'$ when $\mathfrak{m} \mid \mathfrak{m}'$. It generalizes the fact that if $m \mid n$ then the natural map $(\mathbf{Z}/n\mathbf{Z})^\times \to (\mathbf{Z}/m\mathbf{Z})^\times$ is onto.

[10]This technical meaning for Führer was translated as *leader* in the early 1930s. See p. 655 of https://www.ams.org/journals/bull/1931-37-09/S0002-9904-1931-05214-9/S0002-9904-1931-05214-9.pdf.

Example 5.6. The extension $\mathbf{Q}(i)/\mathbf{Q}$ has admissible modulus 4∞, but neither 2∞ nor 4 are admissible for $\mathbf{Q}(i)/\mathbf{Q}$ since $[I_{2\infty} : P_{2\infty}]$ and $[I_4 : P_4]$ equal 1 rather than 2, so the conductor of $\mathbf{Q}(i)/\mathbf{Q}$ is 4∞.

Theorem 5.7 (Takagi, 1920)**.** *Let K be a number field.*

(1) (*Existence*) *To each ideal group H there is a class field over K.*
(2) (*Isomorphism*) *If H is an ideal group with modulus $\mathfrak{m}$ and has class field L/K, then* $\mathrm{Gal}(L/K) \cong I_{\mathfrak{m}}/H$.
(3) (*Completeness*) *Each finite abelian extension of K is a class field.*
(4) (*Comparison*) *If H_1 and H_2 are ideal groups with common modulus $\mathfrak{m}$ and they have class fields L_1 and L_2 (inside a common algebraic closure of K), then $L_1 \subset L_2 \iff H_2 \subset H_1$.*
(5) (*Conductor*) *For every finite abelian extension L/K, the places of K appearing in the conductor $\mathfrak{f}_{L/K}$ are the ramified places for L/K.*
(6) (*Decomposition*) *If H is an ideal group with modulus $\mathfrak{m}$ and class field L/K, then each prime $\mathfrak{p} \nmid \mathfrak{m}$ is unramified in L and the residue field degree $f_{\mathfrak{p}}(L/K)$ equals the order of $\mathfrak{p}$ in $I_{\mathfrak{m}}/H$.*

Some parts of this theorem had been proved earlier by Weber (comparison, and in some cases isomorphism).

Here are two immediate consequences of Takagi's theorem.

- A new proof of the first three parts of Conjecture 4.1 on the Hilbert class field. Taking $\mathfrak{m} = (1)$ and $H = P_{(1)}$, $I_{\mathfrak{m}}/H$ is the ideal class group of K, so the existence, isomorphism, and decomposition theorems imply the first and third parts of Conjecture 4.1. To prove the second part of Conjecture 4.1, let K'/K be a finite abelian extension unramified at all places of K. By the conductor theorem, $\mathfrak{f}_{K'/K} = (1)$, so by the completeness theorem K' is the class field of an ideal group H' such that $P_{(1)} \subset H' \subset I_{(1)}$. Since $P_{(1)} = H$, we get $H \subset H'$, which implies $K' \subset K$ by the comparison theorem. The last part of Conjecture 4.1, that all ideals in K becomes principal in the Hilbert class field of K, does *not* follow from Takagi's work, so it remained an open problem.
- A generalization of the Kronecker–Weber theorem to all number fields K: for each K-modulus $\mathfrak{m}$ there is a finite abelian extension $K_{\mathfrak{m}}$ of K such that every finite abelian extension of K is contained in some $K_{\mathfrak{m}}$. These fields $K_{\mathfrak{m}}$ are defined through the existence theorem: $K_{\mathfrak{m}}$ is the class field over K of the ideal group $P_{\mathfrak{m}}$. This definition uniquely determines $K_{\mathfrak{m}}$ by the comparison theorem (two abelian extensions that are class fields to the same ideal group must be equal). Now if we choose a finite abelian extension L/K, the completeness theorem tells us that there is a K-modulus $\mathfrak{m}$ and ideal group H where $P_{\mathfrak{m}} \subset H \subset I_{\mathfrak{m}}$ such that L is the class field to H. By the comparison theorem, the containment $P_{\mathfrak{m}} \subset H$ implies $L \subset K_{\mathfrak{m}}$, so every finite abelian extension of K lies in one of the fields $K_{\mathfrak{m}}$.

  The fields $K_{\mathfrak{m}}$ are called *ray class fields* over K. For $m \geq 1$, the ray class field over $\mathbf{Q}$ for $\mathfrak{m} = (m\infty)$ is $\mathbf{Q}(\zeta_m)$ and the ray class field over $\mathbf{Q}$ for $\mathfrak{m} = (m)$ is the maximal real subfield $\mathbf{Q}(\zeta_m + \zeta_m^{-1})$ of $\mathbf{Q}(\zeta_m)$, so the ray class fields of $\mathbf{Q}$ are cyclotomic fields and their maximal real subfields. The isomorphism in Theorem 5.7(2) in these two cases says $\mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q}) \cong (\mathbf{Z}/m\mathbf{Z})^\times$ and $\mathrm{Gal}(\mathbf{Q}(\zeta_m + \zeta_m^{-1})/\mathbf{Q}) \cong (\mathbf{Z}/m\mathbf{Z})^\times/\{\pm 1\}$ by Examples 3.3 and 5.3. So ray class fields $K_{\mathfrak{m}}$ for $K \neq \mathbf{Q}$ generalize cyclotomic

extensions of $\mathbf{Q}$, but their definition is very indirect. In particular, the fields $K_{\mathfrak{m}}$ are generally not equal to or contained in cyclotomic extensions of K.

One way to see ray class fields of K don't all lie in cyclotomic extensions of K when $K \neq \mathbf{Q}$ was pointed out in [14]: if $K \neq \mathbf{Q}$ then there are many quadratic (hence abelian) extensions L/K that are not in a cyclotomic extension of K, so a ray class field over K that contains L is not in a cyclotomic extension of K. To create such a quadratic extension, let p be an odd prime number that splits completely in K. (There are infinitely many of these primes, for each K.[11]) Pick a prime $\mathfrak{p}$ of K lying over p. By the Chinese remainder theorem there is an $\alpha \in \mathcal{O}_K$ such that $\alpha \in \mathfrak{p} - \mathfrak{p}^2$ and $\alpha \equiv 1 \bmod \mathfrak{q}$ for each prime $\mathfrak{q}$ in K other than $\mathfrak{p}$ that lies over p. Since $\mathrm{ord}_{\mathfrak{p}}(\alpha) = 1$, α is not a square in K. Set $L := K(\sqrt{\alpha})$, which is quadratic over K. From the way we chose α, the extension L/K is ramified at $\mathfrak{p}$ and unramified at the other primes in K lying over p. Check as an exercise that if $K \subset F \subset K(\zeta_m)$ and some prime in K lying over p is unramified in F then *every* prime in K lying over p is unramified in F. (For this, we just need that p is unramified in K, not that it splits completely.) The field L violates that property, so L is not in a cyclotomic extension of K. We have proved that the naive generalization of the Kronecker-Weber theorem to number fields besides $\mathbf{Q}$ is false: when $K \neq \mathbf{Q}$, there are abelian extensions of K that are not contained in a cyclotomic extension of K, and we can even choose such extensions to be quadratic.

Now we make some comments on the different parts of Takagi's theorem.

Takagi proved the existence theorem from a counting argument, starting with the cyclic case. To this day, all proofs of class field theory use a reduction to the cyclic case. The complicated index calculations Takagi used in this proof were later streamlined by Herbrand.

The isomorphism and completeness theorems say the technically defined class fields over K are the same as the finite abelian extensions of K. Takagi at first didn't believe the completeness theorem was really possible, *i.e.*, that every finite abelian extension is a class field. He wrote that trying to explain why this idea should be wrong almost led him to a nervous breakdown. At that time nobody else in Japan was studying algebraic number theory, so Takagi had no local colleagues who could check his work. Takagi did not prove the isomorphism theorem with an explicit isomorphism, but only obtained it indirectly (finite abelian groups have enough numerical invariants to make this possible, *e.g.*, two cyclic groups are abstractly isomorphic as soon as we know they have the same size). Artin later contributed the essential ingredient to class field theory by writing down a natural and explicit isomorphism from the Galois group to the ideal group.

In Takagi's proof of the completeness theorem, he used (5.1) and an inequality that is its reverse for abelian L/K:

$$[I_{\mathfrak{m}} : H_{\mathfrak{m}}(L/K)] \geq [L : K] \tag{5.2}$$

for some $\mathfrak{m}$. Note (5.1) is valid for all Galois extensions, while (5.2) is stated only for abelian extensions (and in fact is false for all other extensions). Takagi proved (5.2) only for cyclic extensions of prime degree, which sufficed for his inductive proof. Later Hasse found a proof of (5.2) that did not need a restriction to prime degree. Unlike the proof of (5.1), which

[11]We won't really need the full strength of p splitting completely in K. It will be sufficient that p is unramified in K and has at least two primes lying over it in K. Taking it to split completely is the easiest way to make those two properties hold.

uses Weber's L-functions, the proof of (5.2) is purely algebraic and its ideas go back to work of Gauss on quadratic forms.

The comparison theorem resembles Galois theory as long as we focus on class fields with a common admissible modulus $\mathfrak{m}$. (These are the fields between K and the class field over K for $P_{\mathfrak{m}}$.) Their corresponding ideal groups with modulus $\mathfrak{m}$ are the subgroups between $P_{\mathfrak{m}}$ and $I_{\mathfrak{m}}$. However, if we start to consider *all* class fields at once, then we run into a comparison problem: the admissible moduli for two class fields might not be the same, so we have to pass to a common admissible modulus for the two extensions before we can compare them by their ideal groups. This is like comparing two abelian extensions of $\mathbf{Q}$ by Galois theory only after embedding them in a common cyclotomic field, so it's not a far-out idea at all. If we want a bijection between all ideal groups in K and all class fields over K, in the spirit of Galois theory, we need to identify together the ideal groups that have the same class field. When does this happen? If H and H' are ideal groups for K defined with moduli $\mathfrak{m}$ and $\mathfrak{m}'$ (that is, $P_{\mathfrak{m}} \subset H \subset I_{\mathfrak{m}}$ and $P_{\mathfrak{m}'} \subset H \subset I_{\mathfrak{m}'}$), call H and H' *equivalent* if there is a modulus $\mathfrak{m}''$ divisible by both $\mathfrak{m}$ and $\mathfrak{m}'$ such that the natural homomorphisms $I_{\mathfrak{m}''} \to I_{\mathfrak{m}}/H$ and $I_{\mathfrak{m}''} \to I'_{\mathfrak{m}}/H'$ have the *same kernel*, which says $H \cap I_{\mathfrak{m}''} = H' \cap I_{\mathfrak{m}''}$. Two ideal groups in K that are equivalent in this sense have the same class field over K, and the correspondence between class fields over K and ideal groups in K up to equivalence is a bijection. This notion of equivalent ideal groups goes back to Weber, and is awkward. When we pass to the language of ideles later, all equivalent ideal groups will merge into a single subgroup of the ideles, making class field theory simpler.

The conductor theorem suggests the conductor and discriminant of an abelian extension are related, since their prime factors agree.

Theorem 5.8 (Hasse)**.** *Let L/K be abelian and $\mathfrak{m}$ be an admissible modulus for L/K. For a character χ of $I_{\mathfrak{m}}/H_{\mathfrak{m}}$, let L_χ be the class field to $\ker\chi$ and set $\mathfrak{f}_\chi$ to be the conductor of L_χ/K. Then the discriminant of L/K is given by the formulas*

$$\operatorname{disc}(L/K) = \prod_{\chi} \mathfrak{f}_{\chi,f},$$

where χ runs over all characters of $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ and $\mathfrak{f}_{\chi,f}$ is the finite part of $\mathfrak{f}_\chi$.

This is the conductor-discriminant formula (or Führerdiskriminantenproduktformel). It expresses the discriminant of an abelian extension L/K in terms of conductors of cyclic subextensions L_χ/K (the Galois group of L_χ/K is isomorphic to the image of χ, which is a cyclic group). Hasse's proof used complex analysis, specifically the decomposition of $\zeta_L(s)$ into a product of Weber L-functions for the characters of $I_{\mathfrak{m}}/H_{\mathfrak{m}}$. In addition to writing $\operatorname{disc}(L/K)$ as a product of the finite parts of the $\mathfrak{f}_\chi$, Hasse showed the conductor $\mathfrak{f}_{L/K}$ is their least common multiple (retaining the infinite places).

Example 5.9. When $[L : K] = 2$, $\operatorname{disc}(L/K)$ is the finite part of $\mathfrak{f}_{L/K}$. In particular, if $L = \mathbf{Q}(\sqrt{d})$ for squarefree d, the conductor of L has finite part $|d|$ or $4|d|$, which proves $4|d|\infty$ is an admissible modulus for $\mathbf{Q}(\sqrt{d})$.

The decomposition theorem shows a prime $\mathfrak{p} \nmid \mathfrak{m}$ splits in L if and only if $\mathfrak{p} \in H$, so Weber's and Takagi's notions of class field agree. Takagi's definition in terms of norms rather than prime splitting just happens to be more convenient than Weber's as a starting point to prove theorems.

The decomposition theorem also tells us a special property of abelian extensions: the primes that split in the extension are described by "congruence conditions." For instance,

the primes p splitting in $\mathbf{Q}(\sqrt{6})$ satisfy $p \equiv 1, 5, 19, 23 \bmod 24$. (This follows from quadratic reciprocity.) These congruence classes are a subgroup of the units mod 24. To see what this turns into for a general abelian extension L/K, we appeal to the completeness theorem: L is a class field over K for some modulus $\mathfrak{m}$, so a prime $\mathfrak{p}$ of K not dividing $\mathfrak{m}$ splits completely in L if and only if $\mathfrak{p}$ lies in $H_{\mathfrak{m}}$ (since $\mathfrak{p}$ is unramified in L and the order of $\mathfrak{p}$ in $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ is its residue field degree). Therefore the primes not dividing $\mathfrak{m}$ that are in $\mathrm{Spl}(L/K)$ are those in the subgroup $H_{\mathfrak{m}}/P_{\mathfrak{m}}$ of $I_{\mathfrak{m}}/P_{\mathfrak{m}}$, and lying in a subgroup should be thought of as generalized congruence conditions. Since class fields over K are the same thing as abelian extensions, splitting in an abelian extension is described by congruences. Amazingly, the converse is also true by class field theory. To show this, we extend Bauer's theorem.

Lemma 5.10 (Bauer, 1916)**.** *Let L_1 and L_2 be finite extensions of a number field K, with L_2/K Galois. Then $L_1 \subset L_2$ if and only if $\mathrm{Spl}(L_2/K) \subset \mathrm{Spl}(L_1/K)$.*

Proof. Let $\widetilde{L}_1/K$ be the Galois closure of L_1/K. Since L_2/K is Galois, $L_1 \subset L_2$ if and only if $\widetilde{L}_1 \subset L_2$. Also a prime splits completely in an extension if and only if it splits completely in the Galois closure, so $\mathrm{Spl}(L_1/K) = \mathrm{Spl}(\widetilde{L}_1/K)$. Now we can invoke Bauer's theorem given earlier (Theorem 2.6) for a pair of Galois extensions. □

Theorem 5.11. *Let L/K be a finite extension of number fields and assume there is a K-modulus $\mathfrak{m}$ and a finite set S of primes in K containing all $\mathfrak{p} \mid \mathfrak{m}$ such that whether or not a prime $\mathfrak{p} \notin S$ splits in L is determined by the coset of $\mathfrak{p}$ in $I_{\mathfrak{m}}/P_{\mathfrak{m}}$. Then L/K is an abelian extension.*

Proof. Let $R_{\mathfrak{m}}$ be the class field over K of $P_{\mathfrak{m}}$, so a prime in K not dividing $\mathfrak{m}$ splits in $R_{\mathfrak{m}}$ if and only if it is in $P_{\mathfrak{m}}$. Consider the composite extension $LR_{\mathfrak{m}}/K$. The set of primes in K splitting in a finite extension of K is infinite, so there is a prime $\mathfrak{q} \notin S$ that splits in $LR_{\mathfrak{m}}$. Since $\mathfrak{q}$ splits in $R_{\mathfrak{m}}$ and doesn't divide $\mathfrak{m}$, $\mathfrak{q} \in P_{\mathfrak{m}}$. For each prime $\mathfrak{p} \notin S$ that splits in $R_{\mathfrak{m}}$, $\mathfrak{p} \in P_{\mathfrak{m}}$ (by the definition of $R_{\mathfrak{m}}$), so $\mathfrak{p} = \mathfrak{q}$ in $I_{\mathfrak{m}}/P_{\mathfrak{m}}$ and therefore $\mathfrak{p}$ splits in L by hypothesis. Hence the primes of $\mathrm{Spl}(R_{\mathfrak{m}}/K) \subset \mathrm{Spl}(L/K)$ except perhaps for primes dividing $\mathfrak{m}$, so $L \subset R_{\mathfrak{m}}$ by Lemma 5.10. (Here we need that $R_{\mathfrak{m}}/K$ is Galois.) Since $R_{\mathfrak{m}}/K$ is abelian, the subextension L/K is abelian. □

Corollary 5.12. *For a number field $L/\mathbf{Q}$ and $m \in \mathbf{Z}^+$, the following conditions are equivalent:*

1. *for every positive prime p not dividing m, the splitting of p in $L/\mathbf{Q}$ is determined by a congruence condition on $p \bmod m$,*
2. *$L \subset \mathbf{Q}(\zeta_m)$.*

Both Weber and Takagi defined class fields as Galois extensions with a certain additional property (using prime splitting or group indices). In 1929, A. Scholz showed the Galois property can be dropped, as a non-Galois extension L/K has $[I_{\mathfrak{m}} : H_{\mathfrak{m}}(L/K)] < [L : K]$ for all K-moduli $\mathfrak{m}$, so L/K can't be a class field. Thus (5.1) holds for all finite extensions, while (5.2) is still only true for abelian extensions. (The largest size of $[I_{\mathfrak{m}} : H_{\mathfrak{m}}(L/K)]$ as $\mathfrak{m}$ varies is $[\widetilde{L} : K]$, where $\widetilde{L}$ is the maximal abelian extension of K in L.)

At the end of Takagi's paper, he proved Kronecker's Jugendtraum for all imaginary quadratic fields.

6. Canonical Isomorphism (Artin)

References: [12], [15]

With Takagi's class field theory in hand, the next natural step was to search for an analogue for non-abelian Galois extensions. Takagi raised this issue himself when he reported on his work at the 1920 ICM. Artin thought a lot about this problem: what is non-abelian class field theory? He was also thinking about the question of whether $\zeta_K(s)$ "divides" $\zeta_L(s)$ when $K \subset L$, in the sense that the ratio $\zeta_L(s)/\zeta_K(s)$ should be an *entire* function. Hecke showed in 1917 that the zeta-function of each number field is analytic in the complex plane except for a simple pole at $s = 1$, so $\zeta_L(s)/\zeta_K(s)$ is meromorphic on $\mathbf{C}$. The issue is whether the multiplicity of each zero of $\zeta_K(s)$ is bounded above by its multiplicity as a zero of $\zeta_L(s)$ so the ratio of zeta-functions doesn't acquire poles.

Although we can consider here all extensions of numbers fields L/K, the only general theorem that was known was for *abelian* extensions: the ratio $\zeta_L(s)/\zeta_K(s)$ can be expressed as a product of Weber L-functions of nontrivial characters (of an ideal group with class field L/K) and Weber L-functions of nontrivial characters are entire functions, so the ratio $\zeta_L(s)/\zeta_K(s)$ is entire when L/K is abelian. Artin wanted to treat the case when L/K is a non-abelian Galois extension, and in this work he discovered L-functions of representations of Galois groups, which involves Frobenius elements of prime ideals in an essential way. When a Galois group is abelian, its representations are essentially just the (1-dimensionsal) characters of that abelian group and Artin's definition looks like the following.

Definition 6.1 (Artin, 1923)**.** Let L/K be a finite abelian extension with Galois group G. For a character $\chi\colon G \to S^1$ and $\mathrm{Re}(s) > 1$, set

$$L(s,\chi) = \prod_{\mathfrak{p} \text{ unram.}} \frac{1}{1 - \chi(\mathrm{Frob}_{\mathfrak{p}}(L/K))\,\mathrm{N}\mathfrak{p}^{-s}},$$

where the Euler product is taken over the primes of K that are unramified in L.

The Frobenius element of a prime ideal is defined initially for unramified primes in the top field. As a function of unramified primes in the bottom field, Frobenius elements are only well-defined up to conjugation, so in an *abelian* Galois group they are still well-defined elements.

For Artin's L-function to have a clean *functional equation*, which we won't discuss here, there should be Euler factors in the L-function at the ramified primes too. For example, if χ is the trivial character of $\mathrm{Gal}(L/K)$ then $L(s,\chi)$ ought to be $\zeta_K(s)$, but in the above definition Euler factors at ramified $\mathfrak{p}$ are missing. Since there is not a well-defined Frobenius element at ramified primes in a Galois extension, it is not at all clear how to make a correct definition for Euler factors at these primes based on the way Euler factors in the L-function are defined at the unramified primes. In 1923, Artin was able to find the right Euler factors for his L-functions at ramified primes only by a roundabout way *using* class field theory. In 1930 he found a definition of the correct Euler factors at the ramified primes using inertia groups without class field theory.[12] The provisional definition of $L(s,\chi)$ above, using only the unramified primes, will suffice for us.

Here's the situation: Artin created L-functions for characters of (possibly non-abelian) Galois groups of number fields, Weber had L-functions for characters of generalized ideal class groups, and for an *abelian* extension of number fields L/K, Takagi had an isomorphism

$$I_{\mathfrak{m}}/H_{\mathfrak{m}} \cong \mathrm{Gal}(L/K) \tag{6.1}$$

[12]The ramified primes also appear in Artin's definition of the conductor for his L-function, which is a constant in its functional equation.

for all admissible K-moduli $\mathfrak{m}$ (the multiples of the least admissible modulus $\mathfrak{f}_{L/K}$). But Takagi did not find a specific isomorphism between these groups; an isomorphism was only obtained in an indirect way. For the isomorphic groups in (6.1), Weber and Artin had L-functions of their characters, so it was natural to ask for an explicit isomorphism $\varphi\colon I_{\mathfrak{m}}/H_{\mathfrak{m}} \to \mathrm{Gal}(L/K)$ identifying the L-functions: $L_A(s,\chi) = L_W(s,\chi\circ\varphi)$ for every character $\chi\colon \mathrm{Gal}(L/K) \to S^1$, where we L_A and L_W denote the Artin and Weber constructions of L-functions. Recall from (3.1) Weber's L-function of a character $\psi\colon I_{\mathfrak{m}}/H_{\mathfrak{m}} \to S^1$:

$$L(s,\psi) = \prod_{\mathfrak{p}\nmid\mathfrak{m}} \frac{1}{1-\psi(\mathfrak{p})\,\mathrm{N}\mathfrak{p}^{-s}}.$$

Takagi showed each $\mathfrak{p} \nmid \mathfrak{m}$ is unramified in L, and if we use the conductor of L/K as a modulus then the Weber L-function is a product over all unramified primes, just like the Artin L-function. In any event, staring at the Euler factor of $\mathfrak{p}$ in both the Artin and Weber L-functions suggested to Artin an isomorphism φ from $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ to $\mathrm{Gal}(L/K)$: let $\varphi(\mathfrak{p}) = \mathrm{Frob}_{\mathfrak{p}}(L/K)$ for $\mathfrak{p} \nmid \mathfrak{m}$ and extend φ to all of $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ by multiplicativity.

There is certainly no problem in multiplicatively extending a function on prime ideals not dividing $\mathfrak{m}$ to all ideals in $I_{\mathfrak{m}}$, since primes not dividing $\mathfrak{m}$ generate $I_{\mathfrak{m}}$ without multiplicative relations between them, but the catch is whether we truly have a function on $I_{\mathfrak{m}}/H_{\mathfrak{m}}$: if primes $\mathfrak{p}$ and $\mathfrak{q}$ lie in the same coset of $I_{\mathfrak{m}}/H_{\mathfrak{m}}$, is it true that $\mathrm{Frob}_{\mathfrak{p}}(L/K) = \mathrm{Frob}_{\mathfrak{q}}(L/K)$? This is not obvious! In the special case that L/K is abelian of prime degree ℓ and $\mu_\ell \subset K$, Takagi showed a result of this kind in 1922, which must have encouraged Artin that he was on the right track.

Definition 6.2. For an abelian extension L/K and K-modulus $\mathfrak{m}$ divisible by all primes ramifying in L, the *Artin map* $\varphi_{L/K,\mathfrak{m}}\colon I_{\mathfrak{m}} \to \mathrm{Gal}(L/K)$ is given by $\varphi_{L/K,\mathfrak{m}}(\mathfrak{p}) = \mathrm{Frob}_{\mathfrak{p}}(L/K)$ at primes $\mathfrak{p}$ not dividing $\mathfrak{m}$ and extends to $I_{\mathfrak{m}}$ by multiplicativity. For each ideal $\mathfrak{a}$ relatively prime to $\mathfrak{m}$, $\varphi_{L/K,\mathfrak{m}}(\mathfrak{a})$ is called the *Artin symbol* at $\mathfrak{a}$.

Theorem 6.3 (Artin, 1927)**.** *When $\mathfrak{m}$ is a K-modulus divisible by the places of K that ramify in L, the Artin map $\varphi_{L/K,\mathfrak{m}}\colon I_{\mathfrak{m}} \to \mathrm{Gal}(L/K)$ is surjective and its kernel contains $\mathrm{N}_{\mathfrak{m}}(L/K)$. When $\mathfrak{m}$ is admissible for L/K, the kernel of the Artin map is $P_{\mathfrak{m}}\,\mathrm{N}_{\mathfrak{m}}(L/K) = H_{\mathfrak{m}}(L/K)$, so $I_{\mathfrak{m}}/H_{\mathfrak{m}}(L/K) \cong \mathrm{Gal}(L/K)$ by the Artin map.*

This is the *Artin reciprocity law*. The isomorphism in the reciprocity law makes Takagi's isomorphism theorem (which had no specific isomorphism in it) explicit and it also explains the decomposition theorem since $\mathrm{Frob}_{\mathfrak{p}}(L/K)$ has order $f_{\mathfrak{p}}(L/K)$ in $\mathrm{Gal}(L/K)$. Artin conjectured the reciprocity law in 1923, but at the time could only prove it in special cases, such as cyclotomic and Kummer extensions. Several years later, Artin read Chebotarev's field-crossing method with cyclotomic extensions in the proof of the Chebotarev density theorem and used the same idea to prove the reciprocity law in general.

By far the most difficult part of the Artin reciprocity law to prove is that the kernel of $\varphi_{L/K,\mathfrak{m}}$ for admissible $\mathfrak{m}$ contains $P_{\mathfrak{m}}$: for $(\alpha) \in P_{\mathfrak{m}}$, $\varphi_{L/K,\mathfrak{m}}((\alpha)) = 1$.

Example 6.4. For a squarefree integer $d \neq 1$, the Galois group of $\mathbf{Q}(\sqrt{d})/\mathbf{Q}$ has size 2, so it can be (uniquely) identified with $\{\pm 1\}$. By Example 5.9, $4|d|\infty$ is an admissible modulus for $\mathbf{Q}(\sqrt{d})/\mathbf{Q}$. When $p \nmid 4d$, the Artin map $I_{4|d|\infty} \to \mathrm{Gal}(\mathbf{Q}(\sqrt{d})/\mathbf{Q}) \cong \{\pm 1\}$ sends $p\mathbf{Z}$ to $(\frac{d}{p})$, so for $(a, 4d) = 1$ the Artin map sends $a\mathbf{Z}$ to $(\frac{d}{a})$: the Jacobi symbol is a special instance of the Artin map. The Artin reciprocity law in this case says for $a > 0$ with

$a \equiv 1 \bmod 4d$ (a sign condition and a congruence condition) that $(\frac{d}{a}) = 1$. This equality is also a consequence of Jacobi reciprocity, and is nearly equivalent to it.

Example 6.5. We will derive the main law of quadratic reciprocity from Artin reciprocity. For an odd prime p, let $p^* = (-1)^{(p-1)/2}p$, so $p^* \equiv 1 \bmod 4$. (The sign on p^* is chosen so 2 doesn't ramify in $\mathbf{Q}(\sqrt{p^*})$.) The Artin map $I_{p\infty} \to \mathrm{Gal}(\mathbf{Q}(\sqrt{p^*})/\mathbf{Q})$ sends each odd prime ideal $(q) \neq (p)$ to $\mathrm{Frob}_q(\mathbf{Q}(\sqrt{p^*})/\mathbf{Q})$. By Example 5.9, the least admissible $\mathbf{Q}$-modulus for $\mathbf{Q}(\sqrt{p^*})/\mathbf{Q}$ has finite part $|\operatorname{disc}(\mathbf{Q}(\sqrt{p^*})/\mathbf{Q})| = |p^*| = p$, so $p\infty$ is admissible and therefore the kernel of the Artin map contains $P_{p\infty}$ by the Artin reciprocity law.

Identifying $I_{p\infty}/P_{p\infty}$ with $(\mathbf{Z}/p\mathbf{Z})^\times$ and $\mathrm{Gal}(\mathbf{Q}(\sqrt{p*})/\mathbf{Q})$ with $\{\pm 1\}$ makes the Artin map a homomorphism $(\mathbf{Z}/p\mathbf{Z})^\times \to \{\pm 1\}$ with the effect $q \bmod p \mapsto (\frac{p^*}{q})$ for odd (positive) primes $q \neq p$. It's nontrivial since the Artin map is onto. The only homomorphism from $(\mathbf{Z}/p\mathbf{Z})^\times$ onto $\{\pm 1\}$ is the Legendre symbol $(\frac{\cdot}{p})$, so $(\frac{p^*}{q}) = (\frac{q}{p})$. Replacing p^* with $(-1)^{(p-1)/2}p$ and using the formula $(\frac{-1}{q}) = (-1)^{(q-1)/2}$, we get $(-1)^{(p-1)/2\cdot(q-1)/2}(\frac{p}{q}) = (\frac{q}{p})$.

Remark 6.6. Since only p and (perhaps) ∞ ramify in $\mathbf{Q}(\sqrt{p^*})/\mathbf{Q}$, this extension has admissible modulus $p^r\infty$ for some $r \geq 1$. The Artin map $I_{p^r\infty} \to \mathrm{Gal}(\mathbf{Q}(\sqrt{p^*})/\mathbf{Q})$ is trivial on $P_{p^r\infty}$ by Artin reciprocity, so view it as a surjective homomorphism $(\mathbf{Z}/p^r\mathbf{Z})^\times \to \{\pm 1\}$. The group $(\mathbf{Z}/p^r\mathbf{Z})^\times$ is cyclic, so its only homomorphism onto $\{\pm 1\}$ is $a \bmod p^r \mapsto (\frac{a}{p})$ and the proof ends as before: we do not need to know *a priori* that $r = 1$ is possible.

7. Local Class Field Theory (Hasse)

References: [3], [18]

Hasse was interested in class field theory since shortly after his thesis (1923). In the thesis he classified quadratic forms with rational coefficients in terms of the simpler classification of quadratic forms over real and p-adic numbers, expressed concisely as a "local–global principle". His proofs used Dirichlet's theorem on primes and the quadratic reciprocity law in the guise of Hilbert's product formula. Hasse extended this work (1924) to quadratic forms with coefficients in a number field, using Weber's generalization of Dirichlet's theorem and the Hilbert–Furtwängler quadratic reciprocity law in number fields.

In 1923/1924, Hasse gave a course on class field theory. At Hilbert's suggestion, the notes for this course developed into a comprehensive report surveying the whole subject as it was known at that time. The first installment of Hasse's *Klassenkörperbericht* appeared in 1926. This made class field theory much more accessible, but note the year: it was missing the crown jewel of class field theory, Artin's reciprocity law (1927). A second part of Hasse's report came out in 1930, incorporating the reciprocity law and showing how it implies all known power reciprocity laws that had been found up to that time.

One flaw in class field theory as described so far is the tendency to avoid dealing with ramified primes. The groups $I_\mathfrak{m}$ for admissible moduli $\mathfrak{m}$ don't include ramified primes, and Frobenius elements are not well-defined at ramified primes, so there is no way to extend the Artin map to ramified prime ideals. However, the Hilbert-Furtwängler version of quadratic reciprocity with the Hilbert symbol $(a, b)_v$ uses all places, ramified and unramified. Hasse generalized this symbol in his report and obtained a product formula for it. Instead of using a symbol with values that are roots of unity, Hasse's symbol has values in a Galois group.

Definition 7.1. Let L/K be an abelian extension, $\alpha \in K^\times$, and v a place of K. Define $(\alpha, L/K)_v \in \mathrm{Gal}(L/K)$ by the following procedure.[13]

Write $\mathrm{Gal}(L/K) \cong I_\mathfrak{m}/H_\mathfrak{m}$, with $\mathfrak{m}$ an admissible modulus for L/K. When v is finite, choose $\alpha_0 \in K^\times$ such that α_0 is close to α at v and α_0 is close to 1 at the places in $\mathfrak{m}$ (excluding v if v is in $\mathfrak{m}$):

$$\mathrm{ord}_v\left(\frac{\alpha_0}{\alpha} - 1\right) \geq \mathrm{ord}_v(\mathfrak{m}), \quad \mathrm{ord}_w(\alpha_0 - 1) \geq \mathrm{ord}_w(\mathfrak{m}), \quad u(\alpha_0) > 0,$$

where w runs over finite places in $\mathfrak{m}$ (excluding v) and u runs over real places in $\mathfrak{m}$. If v is not in $\mathfrak{m}$, include the additional condition $((\alpha_0/\alpha), v) = 1$. (For instance, take $\mathrm{ord}_v(\alpha_0/\alpha - 1) \geq 1$.) Factoring the fractional ideal (α_0) into a product of prime ideals, let $\mathfrak{a}$ be its v-free part, so $(\mathfrak{a}, \mathfrak{m}) = 1$ from the conditions on α_0. Define[14]

$$(\alpha, L/K)_v = \varphi_{L/K,\mathfrak{m}}(\mathfrak{a})^{-1}. \tag{7.1}$$

For infinite v where K_v is real, L_v is complex, and $\alpha < 0$ in K_v, set $(\alpha, L/K)_v$ to be the complex conjugation in $\mathrm{Gal}(L_v/K_v) \subset \mathrm{Gal}(L/K)$. For other infinite v, set $(\alpha, L/K)_v$ to be the identity.

Of course Hasse needed to check $(\alpha, L/K)_v$ is independent of the choice of α_0: if β_0 has the same properties as α_0, then $(\alpha_0/\beta_0) \in P_\mathfrak{m}$, so $\varphi_{L/K,\mathfrak{m}}((\alpha_0/\beta_0)) = 1$ by the Artin reciprocity law. Therefore Hasse's symbol $(\alpha, L/K)_v$ is well-defined, but we have to bring in some heavy machinery to show it! (The symbol also has to be shown to be independent of the choice of admissible modulus $\mathfrak{m}$, which is fairly straightfoward to do by comparing the construction at $\mathfrak{m}$ with that at the conductor $\mathfrak{f}_{L/K}$.) When $L = K(\sqrt{\beta})$ and we identify $\mathrm{Gal}(L/K)$ with $\{\pm 1\}$, $(\alpha, L/K)_v$ equals the quadratic Hilbert symbol $(\alpha, \beta)_v$, so Hasse's construction generalizes the Hilbert symbol.

Example 7.2. We compute $(-1, \mathbf{Q}(i)/\mathbf{Q})_v$. (The exponent -1 in (7.1) won't matter here, since values of the symbol are their own inverses.) An admissible modulus for $\mathbf{Q}(i)/\mathbf{Q}$ is 4∞. We identify $\mathrm{Gal}(\mathbf{Q}(i)/\mathbf{Q})$ with $\{\pm 1\}$. Since $-1 < 0$ in $\mathbf{R}$, $(-1, \mathbf{Q}(i)/\mathbf{Q})_\infty = -1$. For $p = 2$, we can use $\alpha_0 = 3$, so $(-1, \mathbf{Q}(i)/\mathbf{Q})_2 = \mathrm{Frob}_3(\mathbf{Q}(i)/\mathbf{Q}) = (\frac{-1}{3}) = -1$. For an odd prime p, we can use $\alpha_0 = 1$, so $(-1, \mathbf{Q}(i)/\mathbf{Q})_p = 1$.

Example 7.3. In a similar way, $(3, \mathbf{Q}(i)/\mathbf{Q})_v = -1$ at $v = 2$ and $v = 3$, and $(3, \mathbf{Q}(i)/\mathbf{Q})_v = 1$ at other v (including $v = \infty$).

Example 7.4. Pick a prime $\mathfrak{p}$ not dividing an admissible modulus $\mathfrak{m}$ for L/K. For $\alpha \in K^\times$, let $k = \mathrm{ord}_\mathfrak{p}(\alpha)$. Choose α_0 so that $\mathrm{ord}_w(\alpha_0 - 1) \geq \mathrm{ord}_w(\mathfrak{m})$ for all $w \mid \mathfrak{m}_f$, $w(\alpha_0) > 0$ for all $w \mid \mathfrak{m}_\infty$, and $\mathrm{ord}_\mathfrak{p}(\alpha_0/\alpha - 1) \geq 1$. Then $(\alpha_0) = \mathfrak{p}^k\mathfrak{a}$ where $\mathfrak{a}$ is relatively prime to $\mathfrak{p}$ and to $\mathfrak{m}$. Then $(\alpha, L/K)_\mathfrak{p} = \varphi_{L/K,\mathfrak{m}}(\mathfrak{a})^{-1} = \varphi_{L/K,\mathfrak{m}}((\alpha_0)\mathfrak{p}^{-k})^{-1}$. By Artin reciprocity, (α_0) is in the kernel of the Artin map, so

$$(\alpha, L/K)_\mathfrak{p} = \varphi_{L/K,\mathfrak{m}}(\mathfrak{p})^k = \mathrm{Frob}_\mathfrak{p}(L/K)^{\mathrm{ord}_\mathfrak{p}(\alpha)}.$$

That the exponent on the right is $\mathrm{ord}_\mathfrak{p}(\alpha)$ rather than $-\mathrm{ord}_\mathfrak{p}(\alpha)$ comes from the exponent -1 in the definition of $(\alpha, L/K)_\mathfrak{p}$. We see that $(\alpha, L/K)_\mathfrak{p}$ has a simple definition in terms of Frobenius elements when $\mathfrak{p} \nmid \mathfrak{m}$. In particular, $(\alpha, L/K)_v = 1$ for all but finitely many v since all but finitely many v don't divide $\mathfrak{m}$ and $\mathrm{ord}_\mathfrak{p}(\alpha) = 0$ for all but finitely many $\mathfrak{p}$.

[13]This procedure generalizes an approach of Hilbert to define Hilbert symbols at ramified places. See [22].

[14]Hasse's definition of $(\alpha, L/K)_v$ did not have the exponent -1. We include it to fit normalizations that show up later.

Theorem 7.5 (Hasse, 1930)**.** *For each finite abelian extension of number fields L/K and $\alpha \in K^\times$, $\prod_v (\alpha, L/K)_v = 1$.*

Just as Hasse's definition of $(\alpha, L/K)_v$ depended on the Artin reciprocity law, so too his proof of Theorem 7.5 used the Artin reciprocity law.

Hasse's study of $(\alpha, L/K)_v$ for finite v indicated that it should depend only on the local behavior of K and L at v (that is, on the completion of K at v and of L at a place over v), despite its roundabout global definition in terms of the Artin map at an ideal in K that is relatively prime to v. For example, $(\alpha, L/K)_v$ lies in the common decomposition group $D(w|v)$ for all places $w \mid v$ on L, and this decomposition group is naturally identified with the Galois group of completions $\mathrm{Gal}(L_w/K_v)$. (The definition we gave of $(\alpha, L/K)_v$ for infinite v is directly in terms of the completion at v and it is the generator of the decomposition group $\mathrm{Gal}(L_w/K_v)$ as a subgroup of $\mathrm{Gal}(L/K)$.) This led Hasse to the discovery of class field theory for local fields. The first version of local class field theory was worked out by Hasse and F. K. Schmidt in 1930 and used global class field theory in an essential way: an abelian extension of local fields is realized as the completion of an abelian extension of number fields, and the global Artin map for that extension of number fields is used to define a local Artin map.

Here's how it goes. Starting with an abelian extension E/F of a (characteristic 0) local field F, write $F = K_v$ for some number field K and finite place v on K. (Every local field of characteristic 0 contains a dense number field, so such K and v exist, in many ways in fact.) Takagi's class field theory implies there is an abelian extension L/K such that $E = LK_v$. (It is generally *not* true that one can arrange for $[L : K] = [E : F]$, even when E/F is cyclic. The Grunwald-Wang theorem describes when that is possible.) For $\alpha \in K^\times$, the symbol $(\alpha, L/K)_v$ belongs to $D(w|v)$, which is naturally identified with $\mathrm{Gal}(L_w/K_v) = \mathrm{Gal}(E/F)$. Hasse defined $(\alpha, E/F) \in \mathrm{Gal}(E/F)$ to be the element in $\mathrm{Gal}(E/F)$ corresponding to $(\alpha, L/K)_v$. So we have a function $K^\times \to \mathrm{Gal}(E/F)$ by $\alpha \mapsto (\alpha, E/F)$. This function is a homomorphism and is v-adically locally constant, so it extends to all $\alpha \in K_v^\times = F^\times$, giving a homomorphism $(-, E/F)\colon F^\times \to \mathrm{Gal}(E/F)$ called the *local Artin map*. In particular, if E/F is unramified and π is a prime in F then Example 7.4 implies $(\pi, E/F)$ is the local Frobenius element in $\mathrm{Gal}(E/F)$ (just like the global Artin map associates to an unramified prime ideal its Frobenius element). If we had not used the exponent -1 to define $(-, L/K)_v$ then $(\pi, E/F)$ would be the inverse of the Frobenius when E/F is unramified.

Compatibility properties of the global Artin map show $(-, E/F)$ is independent of the number fields K and L and the place v on K used to construct it. It turns out that $(-, E/F)$ has kernel equal to the norm subgroup $\mathrm{N}_{E/F}(E^\times) \subset F^\times$. This is a local analogue of $\mathrm{N}_{\mathfrak{m}}(L/K)$ being part of the kernel of the global Artin map $\varphi_{L/K,\mathfrak{m}}$, but in the local case the norm subgroup is the full kernel.

Theorem 7.6. *For an abelian extension of local fields E/F with characteristic 0, the local Artin map $\alpha \mapsto (\alpha, E/F)$ is a homomorphism from $F^\times$ onto $\mathrm{Gal}(E/F)$ with kernel $\mathrm{N}_{E/F}(E^\times)$, so $F^\times/\mathrm{N}_{E/F}(E^\times) \cong \mathrm{Gal}(E/F)$. Associating to E the group $\mathrm{N}_{E/F}(E^\times)$ gives a one-to-one inclusion-reversing correspondence between finite abelian extensions of F and subgroups of finite index in $F^\times$.*

The image of $\mathcal{O}_F^\times$ in $\mathrm{Gal}(E/F)$ under the local Artin map is the inertia group $I(E/F)$, so

$$e(E/F) = [\mathcal{O}_F^\times \mathrm{N}_{E/F}(E^\times) : \mathrm{N}_{E/F}(E^\times)] = [\mathcal{O}_F^\times : \mathrm{N}_{E/F}(\mathcal{O}_E^\times)].$$

Then $f(E/F) = \frac{[E:F]}{e(E/F)} = [F^\times : \mathcal{O}_F^\times \mathrm{N}_{E/F}(E^\times)]$ *is the order of* π *in* $F^\times/\mathcal{O}_F^\times \mathrm{N}_{E/F}(E^\times)$ *for each prime* π *of* F.

When E/F *is not abelian,* $[F^\times : \mathrm{N}_{E/F}(E^\times)] < [E : F]$.

If $H \subset F^\times$ is a subgroup of finite index, call E the *class field* to H over F when $\mathrm{N}_{E/F}(E^\times) = H$. Theorem 7.6 shows Takagi's theorems about class fields over number fields have analogues for class fields over local fields. The only missing part is the local analogue of the conductor. For this, we need a local substitute for the ideal groups $P_\mathfrak{m}$. It is the subgroups $U_n = 1 + \pi^n \mathcal{O}_F$ for $n \geq 1$ and $U_0 = \mathcal{O}_F^\times$. Every ideal group in a number field contains some $P_\mathfrak{m}$ (by definition) and every subgroup of $F^\times$ with finite index, say d, contains all dth powers and thus contains some U_n by Hensel's lemma.[15] When a subgroup of $F^\times$ contains some U_n, it contains $U_{n'}$ for all $n' \geq n$, so there is a U_n inside it with minimal $n \geq 0$. Specifically, when E/F is abelian, let $U_n \subset \mathrm{N}_{E/F}(E^\times)$ with n as small as possible. The *conductor* of E/F is defined to be the ideal $\pi^n \mathcal{O}_F$, so the conductor is $\mathcal{O}_F$ if and only if E/F is unramified. When E/F is ramified, its conductor is a proper ideal of $\mathcal{O}_F$.

The global conductor-discriminant formula (Theorem 5.8) has a local analogue:

Theorem 7.7. *Let* E/F *be an abelian extension of local fields with characteristic* 0*. For a character* χ *of* $\mathrm{Gal}(E/F)$*, let* $\mathfrak{f}_\chi$ *be the conductor of the class field to* $\ker \chi$*. Then* $\mathrm{disc}(E/F) = \prod_\chi \mathfrak{f}_\chi$*, where the product runs over all characters of* $\mathrm{Gal}(E/F)$*.*

This theorem from local class field theory helps to compute the conductor in global class field theory. For an abelian extension of number fields L/K, its conductor can be computed locally: for a prime $\mathfrak{p}$ of K and $\mathfrak{P} \mid \mathfrak{p}$ in L, choose the least $n_\mathfrak{p} \geq 0$ such that $1 + \pi_\mathfrak{p}^{n_\mathfrak{p}} \mathcal{O}_\mathfrak{p} \subset \mathrm{N}_{L_\mathfrak{P}/K_\mathfrak{p}}(L_\mathfrak{P}^\times)$. The finite part of $\mathfrak{f}_{L/K}$ is $\prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}}$. (The infinite part is the product of the real places of K that extend to complex places of L.) So a K-modulus $\mathfrak{m}$ is admissible for an abelian extension L/K when, for each prime power $\mathfrak{p}^{e_\mathfrak{p}}$ fully dividing $\mathfrak{m}$, $1 + \pi_\mathfrak{p}^{e_\mathfrak{p}} \mathcal{O}_\mathfrak{p} \subset \mathrm{N}_{L_\mathfrak{P}/K_\mathfrak{p}}(L_\mathfrak{P}^\times)$ and $\mathfrak{m}$ is divisible by all real places of K that ramify in L.

Example 7.8. In Example 5.5, we showed 4∞ is the conductor of $\mathbf{Q}(i)/\mathbf{Q}$ by a global argument using norms. Now we will check 4∞ is the conductor of $\mathbf{Q}(i)/\mathbf{Q}$ by local arguments at each place. Since only 2 and ∞ ramify in $\mathbf{Q}(i)$, the least admissible modulus is $2^r\infty$ for some $r \geq 1$. The least value of r is the least $r \geq 1$ such that $1 + 2^r\mathbf{Z}_2 \subset \mathrm{N}_{\mathbf{Q}_2(i)/\mathbf{Q}_2}(\mathbf{Q}_2(i)^\times)$. This norm group is the group of nonzero sums of two squares in $\mathbf{Q}_2^\times$. Since -1 is not a sum of two squares in $\mathbf{Q}_2$, $r \neq 1$. Each $x \equiv 1 \bmod 4\mathbf{Z}_2$ satisfies $x \equiv 1 \bmod 8\mathbf{Z}_2$ or $x \equiv 5 \bmod 8\mathbf{Z}_2$. In the first case x is a 2-adic square, so also a sum of two squares. In the second case, $x/5$ is a 2-adic square, so x is a sum of two squares since 5 is. Therefore we can use $r = 2$.

Since finite abelian extensions of a local field F of characteristic 0 correspond to finite-index subgroups of $F^\times = \pi^\mathbf{Z} \times \mathcal{O}_F^\times \cong \mathbf{Z} \times \mathcal{O}_F^\times$, $\mathrm{Gal}(F^{ab}/F)$ is the profinite completion of $F^\times \cong \mathbf{Z} \times \mathcal{O}_F^\times$, which is $\widehat{\mathbf{Z}} \times \mathcal{O}_F^\times$.

Example 7.9. Taking $F = \mathbf{Q}_p$, the local Kronecker–Weber theorem says every finite abelian extension of $\mathbf{Q}_p$ is inside a cyclotomic extension of $\mathbf{Q}_p$:

$$\mathbf{Q}_p^{ab} = \bigcup_{n \geq 1} \mathbf{Q}_p(\mu_n) = \bigcup_{(n,p)=1} \mathbf{Q}_p(\mu_n) \cdot \bigcup_{r \geq 1} \mathbf{Q}_p(\mu_{p^r}),$$

[15] This step uses characteristic 0. The pth powers in a characteristic p local field don't contain a neighborhood of 1. The relevant subgroups of $F^\times$ for local class field theory in all characteristics are the open subgroups of finite index, but openness follows from finite index in characteristic 0.

where the first union is the maximal unramified extension of $\mathbf{Q}_p$, whose Galois group over $\mathbf{Q}_p$ is isomorphic to $\widehat{\mathbf{Z}}$. The second union has Galois group $\mathbf{Z}_p^\times$ over $\mathbf{Q}_p$. Therefore $\mathrm{Gal}(\mathbf{Q}_p^{ab}/\mathbf{Q}_p) \cong \widehat{\mathbf{Z}} \times \mathbf{Z}_p^\times$, which agrees with local class field theory.

E. Noether felt that there should be a self-contained derivation of local class field theory, and global class field theory should be derived from local class field theory. F. K. Schmidt (1930) announced a local development of local class field theory for tamely ramified extensions, but he did not publish it. The main problem in building local class field theory is defining a local Artin map. This isn't difficult for an unramified extension, since there is a Frobenius element in the local Galois group just as in the global case at unramified primes (Example 7.4). But a local construction of the local Artin map for ramified abelian extensions of local fields is not at all easy. In 1933, Hasse found a local description of the local Artin map for cyclic extensions, and Chevalley extended this to abelian extensions. Their construction came from developments in noncommutative ring theory, which is surprising since class field theory is about commutative Galois groups and commutative fields. The particular noncommutative rings that matter are cyclic algebras.

The definition of a cyclic algebra goes back to Dickson (1906) and generalizes the construction of Hamilton's quaternions $\mathbf{H}$. Note that

$$\begin{aligned}\mathbf{H} &= \mathbf{R} + \mathbf{R}i + \mathbf{R}j + \mathbf{R}k \\ &= (\mathbf{R} + \mathbf{R}i) + (\mathbf{R} + \mathbf{R}i)j \\ &= \mathbf{C} + \mathbf{C}j,\end{aligned}$$

with $j^2 = -1$ and $jz = \overline{z}j$. These rules tells us how to multiply two quaternions when written in the form $z + wj$ for complex numbers z and w. The structure here involves complex conjugation, acting as an element of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$. Dickson replaced $\mathbf{C}/\mathbf{R}$ with a cyclic extension of fields.

Definition 7.10 (Dickson)**.** Let L/K be a cyclic extension of fields with degree n, $\alpha \in K^\times$, and let σ be a generator of $\mathrm{Gal}(L/K)$. The direct sum

$$L \oplus Lx \oplus Lx^2 \oplus \cdots \oplus Lx^{n-1},$$

where

- $x^n = \alpha$,
- $x\gamma = \sigma(\gamma)x$ for all $\gamma \in L$,

is called a *cyclic algebra* over K.

Here K can be an arbitrary field, not just a number field. There is no obvious link between cyclic algebras and class field theory. Dickson later called these "algebras of type D" but his hint was not followed (unlike Banach's "espaces du type (B)"). Cyclic algebras are not called Dickson algebras.

Example 7.11. Hamilton's quaternions are $(\mathbf{C}/\mathbf{R}, c, -1)$, and $(\mathbf{C}/\mathbf{R}, c, 1) \cong \mathrm{M}_2(\mathbf{R})$.

Example 7.12. The quaternions with rational coefficients equal $(\mathbf{Q}(i)/\mathbf{Q}, c, -1)$. They are also $(\mathbf{Q}(\sqrt{-5})/\mathbf{Q}, c, -6)$, which is not obvious! The same cyclic algebra arising from different cyclic extensions is like different polynomials having the same splitting field.

Theorem 7.13. *With notation as above,*

(1) $(L/K, \sigma, \alpha)$ *has center* K*,* K*-dimension* n^2*, and is a simple* K*-algebra (no 2-sided ideals besides* (0) *and* (1)*),*

(2) $(L/K, \sigma, 1) \cong \mathrm{M}_n(K)$ *as* K*-algebras,*
(3) $(L/K, \sigma, \alpha) \cong (L/K, \sigma, \beta)$ *as* K*-algebras if and only if* $\alpha/\beta \in \mathrm{N}_{L/K}(L^\times)$*,*
(4) *for* $(t, n) = 1$*,* $(L/K, \sigma^t, \alpha) \cong (L/K, \sigma, \alpha^u)$*, where* $tu \equiv 1 \bmod n$*.*

Theorem 7.13(3) resembles the equivalence $(\alpha, E/F) = (\beta, E/F) \iff \alpha/\beta \in \mathrm{N}_{E/F}(E^\times)$ in local class field theory. This suggests (but does not say how!) that cyclic algebras could be a route to a purely local definition of the local Artin symbol $(\alpha, E/F)$, and that is exactly what happened. We need to know what the cyclic algebras over a local field look like. Hasse showed they can always be put into a standard form using *unramified* extensions.

Theorem 7.14 (Hasse, 1931)**.** *Every cyclic algebra over a local field* F *of characteristic* 0 *with* F*-dimension* n^2 *is a cyclic algebra of the form*

$$(F_n/F, \mathrm{Frob}, \pi^a),$$

where F_n *is the unramified extension of* F *with degree* n*,* Frob *is the canonical generator of* $\mathrm{Gal}(F_n/F)$*,* π *is a prime in* F*, and* $a \in \mathbf{Z}$*.*

The norm subgroup $\mathrm{N}_{F_n/F}(F_n^\times)$ from the unramified extension of degree n is $\pi^{n\mathbf{Z}} \times \mathcal{O}_F^\times$, so Theorem 7.13(3) tells us two things:

(1) $(F_n/F, \mathrm{Frob}, \pi^a) \cong (F_n/F, \mathrm{Frob}, \pi^b)$ as F-algebras if and only if $a \equiv b \bmod n$,
(2) $(F_n/F, \mathrm{Frob}, \pi^a)$ is independent of π,

Therefore $a \bmod n$ is a well-defined invariant of the F-algebra $(F_n/F, \mathrm{Frob}, \pi^a)$. By Theorem 7.13, this invariant is $0 \bmod n$ precisely when the algebra is isomorphic to $\mathrm{M}_n(F)$.

Now we can give a local definition of $(\alpha, E/F)$, first for cyclic E/F and then for abelian E/F. Let $n = [E : F]$, with F a local field of characteristic 0. For $\alpha \in F^\times$ and σ a generator of $\mathrm{Gal}(E/F)$, consider the cyclic algebra

$$A = (E/F, \sigma, \alpha).$$

Let the invariant of A be $a \bmod n$. If we change σ, we usually get a new (that is, non-isomorphic) cyclic algebra, so a usually changes. Although A depends on σ, the power

$$\sigma^a \in \mathrm{Gal}(E/F)$$

is independent of σ, and is the local Artin symbol defined before: $(\alpha, E/F) = \sigma^a$.[16] If E/F is abelian, then $E = E_1 \cdots E_r$ for cyclic E_i/F, so $\mathrm{Gal}(E/F)$ embeds into $\prod_i \mathrm{Gal}(E_i/F)$, with its image being sequences that agree on overlaps $E_i \cap E_j$. For $\alpha \in F^\times$, the symbols $(\alpha, E_i/F)$ agree on overlaps $E_i \cap E_j$, so they come by restriction from a single element of $\mathrm{Gal}(E/F)$, which is precisely the local Artin symbol $(\alpha, E/F)$ defined before. By proving *ab ovo* that this construction of $(\alpha, E/F)$ is independent of the cyclic subextensions E_i/F, local class field theory can be developed without global class field theory.

Like Takagi in the global case, this approach to local class field theory does not give an explicit construction of class fields over local fields. Such a construction was achieved later (1965) by Lubin and Tate using formal groups.

We now turn to the case of number fields and get a definition of Hasse's $(\alpha, L/K)_v$ for finite v without recourse to the global Artin map. (For infinite v we already presented a direct local definition of $(\alpha, L/K)_v$.) Let L/K be a cyclic extension of number fields, with degree n. Pick a generator σ of $\mathrm{Gal}(L/K)$. For each $\alpha \in K^\times$, $A := (L/K, \sigma, \alpha)$ is a cyclic

[16] If the definition of a cyclic algebra had $\gamma x = x\sigma(\gamma)$ instead of $x\gamma = \sigma(\gamma)x$ then $(\alpha, E/F) = \sigma^{-a}$.

K-algebra. When v is a finite place of K, the tensor product $K_v \otimes_K A$ is a cyclic K_v-algebra with K_v-dimension n^2.[17] Let it have invariant $a_v \bmod n$. Then $(\alpha, L/K)_v = \sigma^{a_v}$. The definition of $(\alpha, L/K)_v$ for abelian L/K proceeds as in the local case by writing L as a composite of cyclic extensions of K.

Example 7.15. Consider $\mathbf{Q}(i)/\mathbf{Q}$ with Galois group $\{1, c\} \cong \{\pm 1\}$. We want to compute $(-1, \mathbf{Q}(i)/\mathbf{Q})_v$ using the new definition of these symbols. (They were computed before in Example 7.2 with Hasse's original definition.)

Here $n = [\mathbf{Q}(i) : \mathbf{Q}] = 2$ and σ is $c = -1$. The cyclic algebra $(\mathbf{Q}(i)/\mathbf{Q}, c, -1)$ is the rational quaternions $\mathbf{H}(\mathbf{Q})$, and $\mathbf{Q}_v \otimes_{\mathbf{Q}} \mathbf{H}(\mathbf{Q}) \cong \mathbf{H}(\mathbf{Q}_v)$.[18] When $v = \infty$, $\mathbf{H}(\mathbf{Q}_v) = \mathbf{H}(\mathbf{R})$ is the real quaternions, whose invariant is 1 mod 2, so $(-1, \mathbf{Q}(i)/\mathbf{Q})_\infty = c^1 = c = -1$. The 2-adic quaternions are $\mathbf{H}(\mathbf{Q}_2) = (\mathbf{Q}_2(i)/\mathbf{Q}_2, c, -1)$, but this is *not* in the normalized form of Theorem 7.14 for two reasons: $\mathbf{Q}_2(i)/\mathbf{Q}_2$ is ramified and the parameter -1 is not an integral power of 2. It turns out that $\mathbf{H}(\mathbf{Q}_2) \cong (\mathbf{Q}_2(\sqrt{-3})/\mathbf{Q}_2, c, 2)$, which is in standard form, so $a_2 \equiv 1 \bmod 2$. Therefore $(-1, \mathbf{Q}(i)/\mathbf{Q})_2 = c^{a_2} = c = -1$.

For odd primes p, $-1 = x^2 + y^2$ for some x and y in $\mathbf{Q}_p$ (since $-1 \equiv x^2 + y^2 \bmod p$ is solvable and we can lift to a p-adic solution with Hensel's Lemma). Therefore[19] $\mathbf{H}(\mathbf{Q}_p) \cong \mathrm{M}_2(\mathbf{Q}_p)$, whose invariant is 0 mod 2, so $(-1, \mathbf{Q}(i)/\mathbf{Q})_p = 1$.

Armed with the new local definition of $(\alpha, L/K)_v$, Hasse (1933) proved Theorem 7.5 without global class field theory and could derive the Artin reciprocity law from Theorem 7.5. Since Hasse originally used Artin reciprocity to prove Theorem 7.5, Theorem 7.5 is equivalent[20] to the Artin reciprocity law. But unlike Artin reciprocity, all places of K occur in Theorem 7.5, so we are closer to a more balanced formulation of the reciprocity law.

The mathematical structure underlying cyclic algebras (and more general crossed product algebras, which are analogues of cyclic algebras with general Galois extensions in place of cyclic extensions) is group cohomology. After World War II, developments in class field theory led to a stripping away of the algebras (which after all were defined entirely in terms of the number fields themselves) in proofs of class field theory, leaving behind only cohomological formalism. This is how cohomology entered local and global class field theory in the period 1950–1952 in work of Hochschild, Nakayama, Weil, Artin, and Tate.

8. Idelic Class Field Theory (Chevalley)

Reference: [9]

With local class field theory having been set up on its own terms, a remaining task was to derive the theorems of global class field theory from those of local class field theory. The new concept that allowed this is the idele group of a number field. It was first defined by

[17] The base extension of a cyclic algebra to a larger base field might not be a cyclic algebra according to the definition we gave, *e.g.*, $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{M}_2(\mathbf{C})$ apparently isn't a cyclic $\mathbf{C}$-algebra since $\mathbf{C}$ doesn't have a quadratic field extension with which to create a $\mathbf{C}$-algebra of dimension 4. In the definition of a cyclic algebra $(L/K, \sigma, \alpha)$, we could relax the hypothesis that L/K is a cyclic *field* extension to L being a separable K-algebra of dimension n admitting a K-automorphism σ of order n whose fixed set is K. Then, for instance, $\mathrm{M}_n(K) \cong (K^n/K, \sigma, 1)$ where σ is the cyclic shift of the coordinates of K^n. This makes $\mathrm{M}_n(K)$ a cyclic algebra over K whether or not K has a degree n cyclic field extension and the base extension of a cyclic algebra is a cyclic algebra.

[18] For a field K, let $\mathbf{H}(K) = K + Ki + Kj + Kk$ with the usual rules of multiplication on the basis.

[19] For each field K of characteristic not 2, $\mathbf{H}(K) \cong \mathrm{M}_2(K)$ if and only if -1 is a sum of two squares in K.

[20] Proving Theorem 7.5 without class field theory is hard, since it is the Artin reciprocity law in disguise.

Chevalley for the purpose of describing global class field theory for infinite extensions, but several years later he used ideles in a new way to get global class field theory from local class field theory.

Let's see why the classical description of the Artin map, in terms of ideals, is not well-suited to describe infinite abelian extensions of a number field. When $K \subset L \subset L'$ is a tower of finite abelian extensions of K, and a K-modulus $\mathfrak{m}$ is admissible for L' (and thus also for L), the diagram

$$\begin{array}{ccc} & & \mathrm{Gal}(L'/K) \\ & \overset{\varphi_{L'/K,\mathfrak{m}}}{\nearrow} & \big\downarrow \\ I_{\mathfrak{m}} & & \mathrm{res.} \\ & \underset{\varphi_{L/K,\mathfrak{m}}}{\searrow} & \big\downarrow \\ & & \mathrm{Gal}(L/K) \end{array} \tag{8.1}$$

commutes. But as L' grows, the domain $I_{\mathfrak{m}}$ of the Artin map has to change since the modulus needs to become more highly divisible by primes to keep up with the ensuing ramification. (This resembles the slightly annoying feature of Takagi's comparison theorem: ideal groups must be defined to the same K-modulus.) We want to replace $I_{\mathfrak{m}}$ with an object that doesn't involve a modulus, so the behavior of (8.1) as L' grows is cleaner.

Definition 8.1 (Chevalley, 1936)**.** The *idele group* J_K of a number field K is the set of sequences $(x_v)_v$, indexed by the places v of K, such that $x_v \in K_v^\times$ for all v and $x_v \in \mathcal{O}_v^\times$ for all but finitely many v, where $\mathcal{O}_v$ is the ring of integers of K_v.

An element of J_K is called an idele. Chevalley first called it an "élément idéal," abbreviated later (at Hasse's suggestion) to idèle. Under componentwise multiplication, the ideles are a group, and they lie between the direct sum of the $K_v^\times$'s ($x_v = 1$ for all but finitely many v) and the direct product of the $K_v^\times$'s ($x_v \in K_v^\times$ with no constraints). We embed $K^\times \hookrightarrow J_K$ diagonally, the image being called the *principal ideles* (analogue of principal ideals). We also embed $K_v^\times \hookrightarrow J_K$ singly (on the v-coordinate, with 1's elsewhere).

To each idele $\mathbf{x} \in J_K$ we have a fractional ideal

$$\iota(\mathbf{x}) = \prod_{v \nmid \infty} \mathfrak{p}_v^{\mathrm{ord}_v(x_v)},$$

where the right side is a finite product since $x_v \in \mathcal{O}_v^\times$ for all but finitely many v. The image of a principal idele is the principal ideal of the same element of $K^\times$. The archimedean components of $\mathbf{x}$ play no role in $\iota(\mathbf{x})$.

Using this passage from ideles to ideals, each generalized ideal class group of K can be realized as a quotient group of J_K as follows. Pick a K-modulus $\mathfrak{m}$. Starting with an idele $\mathbf{x} \in J_K$, pick $\alpha_0 \in K^\times$ (by the approximation theorem) so that for v in $\mathfrak{m}$ we have

$$\mathrm{ord}_v(x_v/\alpha_0 - 1) \geq \mathrm{ord}_v(\mathfrak{m}) \tag{8.2}$$

when $v \mid \mathfrak{m}_f$, and

$$\frac{x_v}{v(\alpha_0)} > 0 \tag{8.3}$$

when $v \mid \mathfrak{m}_\infty$. (This is the analogue of Hasse's choice of auxiliary α_0 in the definition of the symbol $(\alpha, L/K)_v$.) The idele $\mathbf{x}/\alpha_0 = (\dots, x_v/\alpha_0, \dots)$ has corresponding ideal $\iota(\mathbf{x}/\alpha_0)$ in

$I_\mathfrak{m}$. If $\beta_0 \in K^\times$ has the same properties as α_0 then the ideals $\iota(\mathbf{x}/\alpha_0)$ and $\iota(\mathbf{x}/\beta_0)$ differ by the principal ideal (β_0/α_0), which lies in $P_\mathfrak{m}$, so $\iota(\mathbf{x}/\alpha_0)$ is well-defined in terms of $\mathbf{x}$ as an element of $I_\mathfrak{m}/P_\mathfrak{m}$. Sending $\mathbf{x}$ to $\iota(\mathbf{x}/\alpha_0)$ is a homomorphism from J_K onto $I_\mathfrak{m}/P_\mathfrak{m}$. If $\mathbf{x} = (\alpha, \alpha, \dots)$ is a principal idele, we can use $\alpha_0 = \alpha$, so the image is 1, which means the map $J_K \to I_\mathfrak{m}/P_\mathfrak{m}$ kills all principal ideles. Therefore all generalized ideal class groups of K can be viewed as quotients of the single group J_K, or even of $J_K/K^\times$. If we multiply an archimedean component of $\mathbf{x}$ by a positive real number then the new idele $\mathbf{x}'$ has the same image as $\mathbf{x}$ in $I_\mathfrak{m}/P_\mathfrak{m}$, because $\mathbf{x}$ and $\mathbf{x}'$ admit the same choices for α_0 in (8.2) and (8.3), and $\iota(\mathbf{x}'/\alpha_0) = \iota(\mathbf{x}/\alpha_0)$ since forming fractional ideals from ideles doesn't involve the archimedean components. Therefore generalized ideal class groups are all quotients of $J_K^1/K^\times$, where J_K^1 is the group of ideles with idelic norm 1. (Each idele can be scaled by a positive number in an archimedean component to obtain idelic norm 1, and this scaling doesn't change the image of the idele in a generalized ideal class group.)

As an indication of the simplicity coming from this viewpoint, let's return to the equivalence relation put on ideal groups in K to make the correspondence between class fields and (equivalence classes of) ideal groups a bijection. An ideal group H with modulus $\mathfrak{m}$ can be converted into a subgroup of J_K containing $K^\times$: take the inverse image of $H/P_\mathfrak{m}$ under the map $J_K \to I_\mathfrak{m}/P_\mathfrak{m}$. Two ideal groups H and H' are equivalent (meaning $H \cap I_{\mathfrak{m}''} = H' \cap I_{\mathfrak{m}''}$ for some multiple $\mathfrak{m}''$ of the moduli for H and H') exactly when they correspond to the same group of ideles.

Now we introduce an idelic version of the Artin map. When L/K is an abelian extension of number fields and $\mathfrak{m}$ is an admissible K-modulus for this extension, the composite map

$$\varphi_{L/K} \colon J_K \longrightarrow I_\mathfrak{m}/P_\mathfrak{m} \xrightarrow{\varphi_{L/K,\mathfrak{m}}} \mathrm{Gal}(L/K)$$

is a surjective homomorphism and (by properties of the Artin map $\varphi_{L/K,\mathfrak{m}}$) is *independent* of the choice of admissible $\mathfrak{m}$. This composite map is the *idelic Artin map*, and $\varphi_{L/K}(K^\times) = 1$ from the construction. To describe the full kernel of the idelic Artin map, we need norms on ideles. Define $\mathrm{N}_{L/K} \colon J_L \to J_K$ by $\mathrm{N}_{L/K}(\mathbf{y}) = \mathbf{x}$ where $x_v = \prod_{w|v} \mathrm{N}_{L_w/K_v}(y_w)$ for all places v of K. Then the kernel of $\varphi_{L/K}$ is $K^\times \mathrm{N}_{L/K}(J_L)$, which is an idelic counterpart to $\varphi_{L/K,\mathfrak{m}}$ having kernel $P_\mathfrak{m}\, \mathrm{N}_\mathfrak{m}(L/K)$ for admissible $\mathfrak{m}$ (see Theorem 6.3).

To formulate class field theory as a one-to-one correspondence using ideles, we need a topology on J_K. The topology Chevalley put on J_K was not Hausdorff.[21] It was later replaced by the restricted product topology, where a basic open neighborhood of 1 in J_K is a set $\prod_v U_v$ with U_v an open neighborhood of 1 in $K_v^\times$ for all v and $U_v = \mathcal{O}_v^\times$ for all but finitely many v. (The product topology would have $U_v = K_v^\times$ for all but finitely many v.) With this topology, J_K is a locally compact (Hausdorff) topological group. Using the product topology, J_K would *not* be locally compact, which is why the product topology is not a good choice.

Theorem 8.2. *For an abelian extension of number fields L/K, the idelic Artin map $\mathbf{x} \mapsto \varphi_{L/K}(\mathbf{x})$ is a homomorphism from J_K onto $\mathrm{Gal}(L/K)$ with kernel $K^\times \mathrm{N}_{L/K}(J_L)$, so $J_K/K^\times \mathrm{N}_{L/K}(J_L) \cong \mathrm{Gal}(L/K)$. Associating to L the group $K^\times \mathrm{N}_{L/K}(J_L)$ gives a one-to-one inclusion-reversing correspondence between finite abelian extensions of K and open subgroups of finite index in J_K that contain $K^\times$.*

[21] See https://mathoverflow.net/questions/41253/who-fixed-the-topology-on-ideles.

For each place v of K, pick a place w in L lying over v. The composite map

$$K_v^\times \longrightarrow J_K \xrightarrow{\varphi_{L/K}} \mathrm{Gal}(L/K)$$

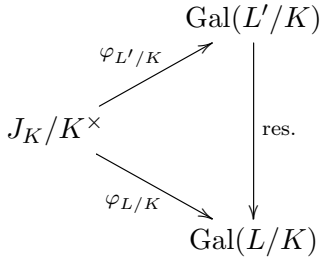
has image the decomposition group $D(w|v)$ and kernel $\mathrm{N}_{L_w/K_v}(L_w^\times)$, and the restriction to $\mathcal{O}_v^\times$ has image the inertia group $I(w|v)$ and kernel $\mathrm{N}_{L_w/K_v}(O_w^\times)$.

Both $D(w|v)$ and $I(w|v)$ do not depend on the choice of w since $\mathrm{Gal}(L/K)$ is abelian.

Using subgroups of $J_K/K^\times$ in Theorem 8.2 instead of subgroups of J_K containing $K^\times$, finite abelian extensions of K correspond one-to-one with "norm subgroups" of $J_K/K^\times$. The isomorphism $J_K/K^\times \mathrm{N}_{L/K}(J_L) \cong \mathrm{Gal}(L/K)$ is analogous to $I_\mathfrak{m}/P_\mathfrak{m}\,\mathrm{N}_\mathfrak{m}(L/K) \cong \mathrm{Gal}(L/K)$.

The idelic class field theory still has a flaw: while the idelic Artin map $\varphi_{L/K}$ is independent of the admissible modulus used in its construction, we are still using an admissible modulus to define it, so a proof of Theorem 8.2 has to fall back on the ideal-theoretic global class field theory. We will see how this flaw gets sorted out below. But first we use the idelic viewpoint to get a workable substitute for (8.1) that lets us pass to infinite abelian extensions.

When $K \subset L \subset L'$ is a tower of finite abelian extension of K and $\mathfrak{m}$ is an admissible K-modulus for L' (and thus also for L), the diagram



commutes and can be used in place of (8.1): the source group $J_K/K^\times$ does not change as L' grows, so we can pass to an inverse limit compatibly to get a homomorphism

$$(-,K)\colon J_K/K^\times \to \mathrm{Gal}(K^{ab}/K) \tag{8.4}$$

mapping to the Galois group of the maximal abelian extension of K. Since the map is onto at finite levels, it has dense image. To show the image is $\mathrm{Gal}(K^{ab}/K)$ consider $(-,K)$ on the subgroup $J_K^1/K^\times$, which is compact (unlike $J_K/K^\times$). Shrinking $J_K/K^\times$ to $J_K^1/K^\times$ maintains surjectivity of the idelic Artin maps $J_K \to \mathrm{Gal}(L/K)$ – this goes back to the fact that each generalized ideal class group is a quotient not just of J_K, but of J_K^1 – so the image of $(-,K)\colon J_K^1/K^\times \to \mathrm{Gal}(K^{ab}/K)$ is dense. The image is also compact, and thus closed, so the image is $\mathrm{Gal}(K^{ab}/K)$. The kernel of (8.4) is the connected component of the identity in $J_K/K^\times$, so $\mathrm{Gal}(K^{ab}/K)$ is the largest totally disconnected quotient group of $J_K/K^\times$.[22]

Finally, we arrive at a description of the idelic Artin map $\varphi_{L/K}$ that doesn't require admissible moduli and illustrates the local-global principle.

Theorem 8.3. *For a finite abelian extension of number fields L/K and $\mathbf{x} \in J_K$,*

$$\varphi_{L/K}(\mathbf{x}) = \prod_v (x_v, L_w/K_v), \tag{8.5}$$

[22]From the viewpoint of infinite Galois theory, $\mathrm{Gal}(K^{ab}/K)$ is the inverse limit of the ideal groups $I_\mathfrak{m}/P_\mathfrak{m}$, partially ordered by reverse divisibility of the K-moduli $\mathfrak{m}$. Weber's equivalence relation on ideal groups, when using Takagi's K-moduli, not just ideals as moduli, says ideal groups are equivalent when they correspond to the same subgroup of the inverse limit of the groups $I_\mathfrak{m}/P_\mathfrak{m}$.

where w is an arbitrary place in L over v and the local Artin symbol $(x_v, L_w/K_v) \in \mathrm{Gal}(L_w/K_v)$ is viewed in $D(w|v)$.

On the right side of (8.5), all but finitely many factors are trivial since for all but finitely many v, L_w/K_v is unramified, $x_v \in \mathcal{O}_v^\times$, and $\mathcal{O}_v^\times \subset \mathrm{N}_{L_w/K_v}(L_w^\times)$ for unramified v. The hard step in the proof of (8.5) is showing the right side is trivial on $K^\times$. This is *exactly* Hasse's old product formula (Theorem 7.5), which is equivalent to the Artin reciprocity law, whose hard step classically was the proof that the global Artin map is trivial on $P_{\mathfrak{m}}$. So we see that all the new notation doesn't make class field theory easier, or change what the hard step is, but the formalism surrounding the difficulties is much more elegant. Reproving Theorem 8.2 by using the right side of (8.5) as a new definition of the idelic Artin map lets global class field theory be derived from local class field theory.

The two classical inequalities (5.1) and (5.2) are still important in the idelic development of class field theory. The idelic version of (5.1) says $[J_K : K^\times \mathrm{N}_{L/K}(J_L)] \leq [L : K]$ for each finite extension L/K, while the idelic version of (5.2) says $[J_K : K^\times \mathrm{N}_{L/K}(J_L)] \geq [L : K]$ when L/K is abelian (and the inequality is false if L/K is not abelian). The original proof of (5.1) used Weber L-functions, so complex analysis was needed. In 1940, Chevalley found a purely algebraic proof of (5.1) in its idelic form. His argument used Pontryagin duality on $J_K/K^\times$ to derive (5.2) in its idelic form without L-functions and using this he could prove (5.1) in its idelic form for abelian extensions. Because of the new logical dependencies, with (5.2) being used to prove (5.1), the order of appearance of the two inequalities in the development of class field theory had to be reversed, which led to a name change. From 1920 to 1940, (5.1) was called the first inequality of class field theory and (5.2) was called the second inequality of class field theory. After 1940, (5.2) was called the first inequality and (5.1) was called the second inequality.

9. Function fields

References: [17], [20]

The development of arithmetic in function fields over finite fields, beyond $\mathbf{F}_q(x)$, began with Artin's 1921 thesis, which explored the arithmetic of quadratic extensions of $\mathbf{F}_q(x)$ and their zeta-functions for odd q. (The thesis itself only treated the case when $q = p$ is prime, but Artin recognized that with no essential changes an arbitrary finite constant field can be used.) Artin used these quadratic extensions to prove a quadratic reciprocity law for $\mathbf{F}_q(x)$ in odd characteristic, adapting an argument of Kummer for the classical quadratic reciprocity law that used parity properties of class numbers of quadratic fields. In 1925, F. K. Schmidt began the development of arithmetic in a general finite extension of $\mathbf{F}_q(x)$, including an (easy) n-th power reciprocity law when the constant field of the extension contains the nth roots of unity. Later Schmidt (1931) sketched a partial development of class field theory for function fields in characteristic p when the degrees of the abelian extensions are *not* divisible by p, following closely the ideas of Takagi. However, Schmidt was not able to treat the existence theorem as Takagi had done for number fields.

In 1934, Hasse proved the Artin reciprocity law[23] in the function field case, using cyclic algebras, as he had done a year before in the number field case. Theorem 7.14 is true for characteristic p local fields by the same argument as for characteristic 0 local fields, leading to a local class field theory in characteristic p. Its theorems are identical to the

[23]Hasse only treated the reciprocity law for cyclic extensions, which is the main case.

characteristic 0 local class field theory, except one needs to be explicit about using *open* subgroups of finite index. Hasse's product formula (Theorem 7.5) is true in the function field case, where it turns out to be a consequence of the residue theorem for function fields. In 1935, Witt proved the existence theorem (for the first time) for abelian extensions with degree divisible by p, complementing Schmidt's work for abelian extensions with degree not divisible by p. This completed the basic statements of class field theory for function fields.

Chevalley's idelic viewpoint makes sense for both number fields and function fields, so it permits a simultaneous development of both cases. However, a dichotomy between the two cases occurs in class field theory for *infinite* abelian extensions. For a function field K, as for a number field, the idelic Artin map $J_K/K^\times \to \mathrm{Gal}(K^{ab}/K)$ has a dense image, but now the map is injective rather than surjective.[24] Its image can be characterized as the elements of $\mathrm{Gal}(K^{ab}/K)$ that, on the algebraic closure of the constant field of K, are integral powers of the Frobenius automorphism.

In the 1950s, Lang developed class field theory for function fields over a finite field using algebraic geometry, at first for unramified abelian extensions and then for ramified abelian extensions. He proved an analogue of Artin's reciprocity isomorphism geometrically without the intervention of inequalities like (5.1) and (5.2). Lang used Rosenlicht's generalized Jacobian varieties (an analogue of Weber's generalized ideal class groups) to show the abelian extensions he described geometrically, together with constant field extensions, account for all finite abelian extensions in the function field case.

These approaches to class field theory over function fields do not give explicit class fields cofinal in all finite abelian extensions, which would be analogous to the cyclotomic fields over $\mathbf{Q}$ (Kronecker–Weber theorem). Even in the number field case, the explicit construction of class fields is still a challenging problem except over $\mathbf{Q}$ and imaginary quadratic fields. Hayes (1974), building on work of Carlitz (1938), constructed an explicit class field theory over rational function fields. That is, Hayes wrote down explicit finite abelian extensions of $\mathbf{F}_q(x)$ such that every finite abelian extension of $\mathbf{F}_q(x)$ is inside one of them.[25] Drinfeld (1974) used Drinfeld modules (called "elliptic modules" by him, from the analogy with complex multiplication on elliptic curves) to achieve the same goal for each function field in one variable over a finite field, not just the rational function field. Roughly, what makes the function field case different from the number field case from the viewpoint of explicit class field theory is that in characteristic p there are far more additive functions.

Appendix A. Second viewpoint on $P_{\mathfrak{m}}$

For a K-modulus $\mathfrak{m}$ in the sense of Definition 5.2, the group $P_{\mathfrak{m}}$ consists of principal fractional ideals with some generator α/β where the numerator and denominator α and β in $\mathcal{O}_K$ satisfy certain properties. We will show below (Corollary A.5) that the properties of α and β can be described entirely in terms of the ratio α/β.

Lemma A.1. *Let $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ be a nonempty finite set of prime ideals. For $\gamma \in K$ such that $\mathrm{ord}_{\mathfrak{p}_j}(\gamma) = 0$ for $j = 1, \dots, r$, we can write $\gamma = \alpha/\beta$ where $\alpha, \beta \in \mathcal{O}_K$ and the principal ideals (α) and (β) each have $\mathrm{ord}_{\mathfrak{p}_j}(\alpha) = 0$ and $\mathrm{ord}_{\mathfrak{p}_j}(\beta) = 0$ for $j = 1, \dots, r$.*

[24] The proof of surjectivity in the number field case breaks down in the function field case because there is nothing like the archimedean places where we can scale an idele to have norm 1 *without* changing its image under the Artin map.

[25] To verify his explicit class field theory worked, Hayes used the abstract class field theory.

Proof. Every element of K is a ratio of algebraic integers. Set $D_\gamma = \{\beta \in \mathcal{O}_K : \beta\gamma \in \mathcal{O}_K\}$, which is the set of all possible denominators for γ along with 0. It's straightforward to check that D_γ is a nonzero ideal in $\mathcal{O}_K$, and this also follows from the formula $D_\gamma = \mathcal{O}_K \cap (1/\gamma)\mathcal{O}_K$.

Claim: for a nonzero prime $\mathfrak{p}$ in $\mathcal{O}_K$, if $\mathfrak{p} \mid D_\gamma$ then $\mathrm{ord}_\mathfrak{p}(\gamma) < 0$.

This claim is analogous to the intuitive fact that if a rational number r written in reduced form has a prime factor p of its denominator, then $\mathrm{ord}_p(r) < 0$. If $h(K) > 1$, then elements of K like γ may not have a reduced form ratio, so the intuition can't be used, but the claim is correct.

Proof of claim: Write $\gamma = a/b$ for $a, b \in \mathcal{O}_K - \{0\}$. Suppose a prime $\mathfrak{p}$ divides D_γ. Then

$$\mathfrak{p} \mid D_\gamma \Longrightarrow D_\gamma \subset \mathfrak{p} \Longrightarrow \mathcal{O}_K \cap \frac{b}{a}\mathcal{O}_K \subset \mathfrak{p} \Longrightarrow a\mathcal{O}_K \cap b\mathcal{O}_K \subset \mathfrak{p}(a),$$

so $\mathfrak{p}(a) \mid \mathrm{lcm}((a),(b))$. Comparing the $\mathfrak{p}$-multiplicity on both sides of that divisibility relation, $1 + \mathrm{ord}_\mathfrak{p}(a) \leq \max(\mathrm{ord}_\mathfrak{p}(a), \mathrm{ord}_\mathfrak{p}(b))$. The maximum can't equal $\mathrm{ord}_\mathfrak{p}(a)$, so $\mathrm{ord}_\mathfrak{p}(a) < \mathrm{ord}_\mathfrak{p}(b)$. Thus $\mathrm{ord}_\mathfrak{p}(\gamma) = \mathrm{ord}_\mathfrak{p}(a) - \mathrm{ord}_\mathfrak{p}(b) < 0$. That proves the claim.

By the claim, since $\mathrm{ord}_{\mathfrak{p}_j}(\gamma) = 0$ for each $\mathfrak{p}_j$ dividing $\mathfrak{m}_f$, we have $\mathfrak{p}_j \nmid D_\gamma$. Therefore D_γ is relatively prime to $\mathfrak{p}_1 \cdots \mathfrak{p}_r$, so $D_\gamma + \mathfrak{p}_1 \cdots \mathfrak{p}_r = (1)$. That means

$$\beta + \delta = 1 \tag{A.1}$$

for some $\beta \in D_\gamma$ and $\delta \in \mathfrak{p}_1 \cdots \mathfrak{p}_r$. We can't have $\beta = 0$ since δ can't be 1. Set $\alpha = \beta\gamma \in \mathcal{O}_K$, so $\alpha \neq 0$ and $\gamma = \alpha/\beta$. Reducing (A.1) modulo $\mathfrak{p}_j$, $\beta \equiv 1 \bmod \mathfrak{p}_j$, so $\mathrm{ord}_{\mathfrak{p}_j}(\beta) = 0$. Also $\mathrm{ord}_{\mathfrak{p}_j}(\gamma) = 0$ by hypothesis, so $\mathrm{ord}_{\mathfrak{p}_j}(\alpha) = \mathrm{ord}_{\mathfrak{p}_j}(\beta\gamma) = 0 + 0 = 0$. □

Example A.2. Let $K = \mathbf{Q}(\sqrt{-5})$ and $\gamma = (1+\sqrt{-5})/(1-\sqrt{-5})$. The prime ideal factorizations of $(1+\sqrt{-5})$ and $(1-\sqrt{-5})$ are $\mathfrak{p}_2\mathfrak{p}_3$ and $\mathfrak{p}_2\mathfrak{p}_3'$, where $\mathfrak{p}_2 = (2, 1+\sqrt{-5})$, $\mathfrak{p}_3 = (3, 1+\sqrt{-5})$, and $\mathfrak{p}_3' = (3, 1-\sqrt{-5})$, so $\gamma\mathcal{O}_K = \mathfrak{p}_2\mathfrak{p}_3/\mathfrak{p}_2\mathfrak{p}_3' = \mathfrak{p}_3\mathfrak{p}_3'^{-1}$. That means $\mathrm{ord}_{\mathfrak{p}_2}(\gamma) = 0$, so there's a way to write γ as α/β where $\alpha, \beta \in \mathcal{O}_K$ and $\mathfrak{p}_2$ doesn't divide (α) or (β). This isn't true for the initial definition of γ as a ratio, since $(1+\sqrt{-5})$ and $(1-\sqrt{-5})$ are both divisible by $\mathfrak{p}_2$. But

$$\gamma = \frac{1+\sqrt{-5}}{1-\sqrt{-5}} = \frac{(1+\sqrt{-5})^2}{(1-\sqrt{-5})(1+\sqrt{-5})} = \frac{-4+2\sqrt{-5}}{6} = \frac{-2+\sqrt{-5}}{3},$$

and the ideals $(-2+\sqrt{-5})$ and (3) are not divisible by $\mathfrak{p}_2$ (their norms are both 9).

Theorem A.3. *Let $\mathfrak{m} = \mathfrak{m}_f\mathfrak{m}_\infty$ be a K-modulus. For $\gamma \in K^\times$, the following conditions are equivalent:*

1. *γ can be written as a ratio α/β for some α and β in $\mathcal{O}_K$ such that both are relatively prime to $\mathfrak{m}_f$, $\alpha \equiv \beta \bmod \mathfrak{m}_f$, and $v(\gamma) > 0$ for all real embeddings v dividing $\mathfrak{m}_\infty$.*
2. *for each prime $\mathfrak{p}$ dividing $\mathfrak{m}_f$, $\mathrm{ord}_\mathfrak{p}(\gamma - 1) \geq \mathrm{ord}_\mathfrak{p}(\mathfrak{m}_f)$ for each $\mathfrak{p}$ dividing $\mathfrak{m}_f$ and $v(\gamma) > 0$ for all real embeddings v dividing $\mathfrak{m}_\infty$.*

Proof. (1) ⇒ (2): When $\mathfrak{p} \mid \mathfrak{m}_f$, We have $\gamma - 1 = (\alpha - \beta)/\beta$, with $\mathfrak{m}_f \mid (\alpha - \beta)$ and $\mathfrak{m}_f$ is relatively prime to (β), so for each prime $\mathfrak{p}$ dividing $\mathfrak{m}_f$,

$$\mathrm{ord}_\mathfrak{p}(\gamma - 1) = \mathrm{ord}_\mathfrak{p}(\alpha - \beta) - \mathrm{ord}_\mathfrak{p}(\beta) = \mathrm{ord}_\mathfrak{p}(\alpha - \beta) \geq \mathrm{ord}_\mathfrak{p}(\mathfrak{m}_f).$$

(2) ⇒ (1): When $\mathfrak{p} \mid \mathfrak{m}_f$, $\mathrm{ord}_\mathfrak{p}(\gamma - 1) \geq \mathrm{ord}_\mathfrak{p}(\mathfrak{m}_f) \geq 1 > 0 = \mathrm{ord}_\mathfrak{p}(1)$, so $\mathrm{ord}_\mathfrak{p}(\gamma) = \mathrm{ord}_\mathfrak{p}(\gamma - 1 + 1) = 0$.

Letting S be the finite set of prime ideals $\mathfrak{p}$ dividing $\mathfrak{m}_f$, we have $\mathrm{ord}_\mathfrak{p}(\gamma) = 0$ for each $\mathfrak{p} \in S$, so Lemma A.1 tells us $\gamma = \alpha/\beta$ where $\alpha, \beta \in \mathcal{O}_K$ and the principal ideals (α) and

(β) satisy $\mathrm{ord}_{\mathfrak{p}}(\alpha) = 0$ and $\mathrm{ord}_{\mathfrak{p}}(\beta) = 0$ for each $\mathfrak{p} \in S$. Thus (α) and (β) are relatively prime to $\mathfrak{m}_f$. When $\mathfrak{p} \mid \mathfrak{m}_f$, the assumption of (2) tells us

$$\mathrm{ord}_{\mathfrak{p}}(\gamma - 1) = \mathrm{ord}_{\mathfrak{p}}(\alpha - \beta) - \mathrm{ord}_{\mathfrak{p}}(\beta) = \mathrm{ord}_{\mathfrak{p}}(\alpha - \beta) \geq \mathrm{ord}_{\mathfrak{p}}(\mathfrak{m}_f),$$

so each prime ideal dividing $\mathfrak{m}_f$ divides $(\alpha - \beta)$ at least as much as it divides $\mathfrak{m}_f$. Therefore $\mathfrak{m}_f \mid (\alpha - \beta)$, so $\alpha \equiv \beta \bmod \mathfrak{m}_f$. □

Remark A.4. We can refine the property $\alpha \equiv \beta \bmod \mathfrak{m}_f$ in Theorem A.3(1) to say that $\alpha, \beta \equiv 1 \bmod \mathfrak{m}_f$. In $(\mathcal{O}_K/\mathfrak{m}_f)^\times$, let the inverse of α mod $\mathfrak{m}_f$ be δ mod $\mathfrak{m}_f$ for some $\delta \in \mathcal{O}_K$. Then δ is relatively prime to $\mathfrak{m}_f$ and $\gamma = (\alpha\delta)/(\beta\delta)$, so renaming $\alpha\delta$ as α and $\beta\delta$ as β makes $\gamma = \alpha/\beta$ where $\alpha \equiv 1 \bmod \mathfrak{m}_f$ and $\beta \equiv 1 \bmod \mathfrak{m}_f$.

Corollary A.5. *For a K-modulus $\mathfrak{m} = \mathfrak{m}_f\mathfrak{m}_\infty$, a principal fractional ideal is in $P_{\mathfrak{m}}$ if and only if it has some generator γ where $\mathrm{ord}_{\mathfrak{p}}(\gamma - 1) \geq \mathrm{ord}_{\mathfrak{p}}(\mathfrak{m}_f)$ for each $\mathfrak{p}$ dividing $\mathfrak{m}_f$ and $v(\gamma) > 0$ for all real embeddings v dividing $\mathfrak{m}_\infty$.*

Proof. By definition, a principal fractional ideal in $P_{\mathfrak{m}}$ has some generator γ fitting condition (1) in Theorem A.3, and that theorem tells us γ also satisfies condition (2), which is the properties we want.

Conversely, assume a principal fractional ideal has a generator γ such that $\mathrm{ord}_{\mathfrak{p}}(\gamma - 1) \geq \mathrm{ord}_{\mathfrak{p}}(\mathfrak{m}_f)$ for each $\mathfrak{p}$ dividing $\mathfrak{m}_f$ and $v(\gamma) > 0$ for all real embeddings v dividing $\mathfrak{m}_\infty$. Then Theorem A.3 says $\gamma = \alpha/\beta$ for some α and β in $\mathcal{O}_K$ such that (α) and (β) are relatively prime to $\mathfrak{m}_f$ and $\alpha \equiv \beta \bmod \mathfrak{m}_f$. Then $(\gamma) \in P_{\mathfrak{m}}$. □

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