Look at the power series for $\sqrt{1+x}$, $\sqrt[3]{1+x}$, and $\sqrt[4]{1+x}$ at $x = 0$:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \cdots,$$

$$\sqrt[3]{1+x} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \frac{22}{729}x^5 - \frac{154}{6561}x^6 + \cdots,$$

$$\sqrt[4]{1+x} = 1 + \frac{1}{6}x - \frac{5}{72}x^2 + \frac{55}{1296}x^3 - \frac{935}{31104}x^4 + \frac{4301}{186624}x^5 - \frac{124729}{6718464}x^6 + \cdots.$$  

It appears that the Taylor coefficients are all rational. Now look at their denominators. In $\sqrt{1+x}$ each coefficient has denominator that is a power of 2, in $\sqrt[3]{1+x}$ each coefficient has denominator that is a power of 3 ($243 = 3^5$ and $729 = 3^6$), and in $\sqrt[4]{1+x}$ each coefficient is a power of 2 times a power of 3 ($1296 = 2^43^4$, $31104 = 2^73^5$, $186624 = 2^83^6$, and $6718464 = 2^{10}3^8$). We will show the power series of $\sqrt{1+x}$ at $x = 0$ has rational Taylor coefficients and the prime factors of the denominators of the coefficients divide $n$.

It is not hard to get a formula for the coefficients in $\sqrt{1+x} = (1+x)^{1/n}$: the coefficient of $x^k$ is

$$\frac{d^k}{dx^k}(1+x)^{1/n} = \frac{1}{n} \left( \frac{1}{n-1} \right) \left( \frac{1}{n-2} \right) \cdots \left( \frac{1}{n-(k-1)} \right) (1+x)^{1/n-k},$$

so the coefficient of $x^k$ is

$$\left. \frac{(\sqrt{1+x})^{(k)}}{k!} \right|_{x=0} = \frac{(1/n)(1/n-1)\cdots(1/n-(k-1))}{k!} = \binom{1/n}{k}.$$

These numbers for $k \geq 0$ are all rational. They are binomial coefficients evaluated at $1/n$.

For $r \in \mathbb{Q}$ the power series for $(1+x)^r$ at $x = 0$ has coefficients that are binomial coefficients evaluated at $r$:

$$(1+x)^r = \sum_{k \geq 0} \frac{r(r-1)(r-2)\cdots(r-(k-1))}{k!} x^k = \sum_{k \geq 0} \binom{r}{k} x^k.$$  

What primes occur in the denominators of $\binom{r}{k}$? Writing $r = a/b$ in reduced form,

$$\binom{a/b}{k} = \frac{(a/b)(a/b-1)(a/b-2)\cdots(a/b-(k-1))}{k!} = \frac{a(a-b)(a-2b)\cdots(a-kb)}{b^k k!}$$

and it is obvious that the only possible primes in the denominator are prime factors of $b$ or a prime factor of $k$!. It is true, but not at all clear, that only prime factors of $b$ matter: a prime factor of $k$! that is not a factor of $b$ gets completely canceled out when the ratio in $\binom{r}{k}$ is simplified. We will explain this purely algebraic phenomenon by using $p$-adic limits!

**Theorem 1.** For rational $r$, a prime dividing the denominator of $\binom{r}{k}$ must divide the denominator of $r$. In particular, a prime dividing the denominator of $\binom{1/n}{k}$ must divide $n$.  

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Proof. To prove the theorem we will prove the contrapositive: each prime $p$ that does not divide the denominator of $r$ also does not divide the denominator of any ${r \choose k}$. Expressed in terms of $p$-adic absolute values, this says: if $|r|_p \leq 1$ then $|{r \choose k}|_p \leq 1$ for $k \geq 0$. To prove this, observe that the expression
\[
\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-(k-1))}{k!}
\]
is a polynomial in $x$ with rational coefficients, so it is a continuous function $\mathbb{Q}_p \to \mathbb{Q}_p$ just as much as it is a continuous function $\mathbb{R} \to \mathbb{R}$ (addition, multiplication, and division in a field are all continuous for any absolute value on the field). When $|r|_p \leq 1$, $r$ lies in $\mathbb{Z}_p$ so $r$ is a $p$-adic limit of nonnegative integers (use the truncations of the $p$-adic expansion of $r$): write $r = \lim_{i \to \infty} m_i$ where $m_i \in \mathbb{N}$. By $p$-adic continuity of polynomials,
\[
\binom{r}{k} = \lim_{i \to \infty} \binom{m_i}{k}.
\]
Each $\binom{m_i}{k}$ is in $\mathbb{Z}$ since binomial coefficients with nonnegative integers upstairs are integers ($\binom{m_i}{k} = 0$ if $0 \leq m_i < k$ and $\binom{m_i}{k} \in \mathbb{Z}^+$ if $m_i \geq k$, by combinatorics). Thus $\binom{m_i}{k} \in \mathbb{Z}_p$ for each $i$, so a $p$-adic limit of $\binom{m_i}{k}$ is in $\mathbb{Z}_p$ ($\mathbb{Z}_p$ is closed in $\mathbb{Q}_p$). Thus $|\binom{r}{k}|_p \leq 1$. \hfill $\square$

Example 2. When the binomial coefficient $\binom{33/20}{7}$ is expanded out and simplified, the denominator can only have prime factors 2 and 5. Explicitly,
\[
\binom{33/20}{7} = -\frac{352590381}{102400000000} = -\frac{352590381}{2^{18}5^8}.
\]

The theorem we proved admits a converse.

Theorem 3. Each prime $p$ that divides the denominator of $r$ also divides the denominator of every $\binom{r}{k}$ for $k \geq 1$.

Obviously this doesn’t hold at $k = 0$, since $\binom{r}{0} = 1$.

Proof. Let’s reformulate the theorem in terms of $p$-adic absolute values: if $|r|_p > 1$ then $|\binom{r}{k}|_p > 1$ for all $k \geq 1$. The top of $\binom{r}{k}$ is $r(r-1)\cdots(r-(k-1))$, and for each positive integer $j$ we have $|r - j|_p = |r|_p$ since $|r|_p > 1 \geq |j|_p$ so
\[
|\binom{r}{k}|_p = \frac{|r(r-1)(r-2)\cdots(r-(k-1))|}{k!}
\]
\[
= \frac{|r|_p|r-1|_p|r-2|_p\cdots|r-(k-1)|_p}{|k!|_p}
\]
\[
= \frac{|r|_p^k}{|k!|_p}
\]
\[
= |r|_p^k p^{\ord_p(k!)}
\]
\[
\geq |r|_p^k
\]
\[
> 1.
\]

\hfill $\square$

Example 4. Every Taylor coefficient of $\sqrt{1+x}$ besides the constant term must have its reduced form denominator divisible by 2 and by 3; it can never be a power of just one of those primes.