In this note, we calculate all the basic invariants of the number field
\[ K = \mathbb{Q}(\sqrt[3]{6}, \omega), \]
where \( \omega = (-1 + \sqrt{-3})/2 \) is a primitive cube root of unity.

Here is the notation for the fields and Galois groups to be used. Let
\[ k = \mathbb{Q}(\sqrt[3]{6}), \]
\[ K = \mathbb{Q}(\sqrt[3]{6}, \omega), \]
\[ F = \mathbb{Q}(\omega) = \mathbb{Q}(\sqrt{-3}), \]
\[ G = \text{Gal}(K/\mathbb{Q}) \cong S_3, \]
\[ N = \text{Gal}(K/F) \cong A_3, \]
\[ H = \text{Gal}(K/k). \]

First we work out the basic invariants for the fields \( F \) and \( k \).

**Theorem 1.** The field \( F = \mathbb{Q}(\omega) \) has ring of integers \( \mathbb{Z}[\omega] \), class number 1, discriminant \( -3 \), and unit group \( \{ \pm 1, \pm \omega, \pm \omega^2 \} \). The ramified prime 3 factors as \( 3 = \sqrt[3]{-3} \). For \( p \neq 3 \), the way \( p \) factors in \( \mathbb{Z}[\omega] = \mathbb{Z}[X]/(X^2 + X + 1) \) is identical to the way \( X^2 + X + 1 \) factors mod \( p \), so \( p \) splits if \( p \equiv 1 \mod 3 \) and \( p \) stays prime if \( p \equiv 2 \mod 3 \).

We now turn to the field \( k \).

Since \( \text{disc}(\mathbb{Z}[\sqrt[3]{6}]) = \text{N}_{k/\mathbb{Q}}(3\sqrt[3]{6})^2 = -36^2 \), only 2 and 3 can ramify in \( k \). Since \( X^3 - 6 \) is Eisenstein at 2 and 3, both 2 and 3 are totally ramified: \( (2) = p_2^3 \), \( (3) = p_3^3 \). So \( \mathcal{O}_k = \mathbb{Z}[\sqrt[3]{6}] \) and \( \text{disc}(\mathcal{O}_k) = -2^23^5 \).

The Minkowski bound on \( k \) is
\[ \frac{3!}{3^3} \left( \frac{4}{\pi} \right) 2 \cdot 3^2\sqrt{3} \approx 8.82. \]

So we want to factor the primes 2, 3, 5, 7. We already know 2 and 3 are totally ramified. Mod 5, \( X^3 - 6 \equiv (X - 1)(X^2 + X + 1) \), so \( (5) = p_5p_5' \), where \( \text{N} p_5 = 5, \text{N} p_5' = 25 \). Since \( X^3 - 6 \equiv (X + 1)(X + 2)(X - 3) \mod 7 \), 7 splits completely.

The norm form for \( k \) is
\[ N_{k/\mathbb{Q}}(a + b\sqrt[3]{6} + \sqrt[3]{36}) = a^3 + 6b^3 + 36c^3 - 18abc, \]
so
\[ (1 + \sqrt[3]{6}) = p_7, \quad (-1 + \sqrt[3]{6}) = p_5, \quad (2 + \sqrt[3]{6}) = p_2p_2', \quad (2 - \sqrt[3]{6}) = p_2, \]
\[ (1 + 2\sqrt[3]{6}) = (p_2')^2, \quad (3 - \sqrt[3]{6}) = p_3p_3', \quad (4 + \sqrt[3]{6}) = p_2p_5p_7', \quad (-3 + \sqrt[3]{36}) = p_3^2. \]

Therefore all prime factors of 2, 3, 5, 7 are principal, so \( h(k) = 1 \). The ratio
\[ \frac{(2 - \sqrt[3]{6})^3}{2} = 1 - 6\sqrt[3]{6} + 3\sqrt[3]{36} \approx .003 \]
is a unit, and its reciprocal is

\[ u \overset{\text{def}}{=} 109 + 60 \sqrt[3]{6} + 33 \sqrt[3]{36} \approx 326.99. \]

The minimal polynomial of \( 2 - \sqrt[3]{6} \) is \( T^3 + 6T^2 + 12T - 2 \), while the minimal polynomial of \( u \) is \( T^3 - 327T^2 + 3T - 1 \).

Other explicit principal ideals also give rise to \( u \). For instance, \( \mathfrak{p}_3 \) is generated by

\[ 3 - 3 + \sqrt[3]{6} \]

which has norm 3, and we get a unit \( > 1 \) from

\[ \frac{(3 + 2 \sqrt[3]{6} + \sqrt[3]{36})^3}{3} = 109 + 60 \sqrt[3]{6} + 33 \sqrt[3]{36}. \]

The ideal \( \mathfrak{p}_7'' \) is generated by

\[ 3 - \sqrt[3]{6} \]

and also by

\[ \frac{4 + \sqrt[3]{6}}{(2 - \sqrt[3]{6})(-1 + \sqrt[3]{6})} = 13 + 7 \sqrt[3]{6} + 4 \sqrt[3]{36} \approx 38.9 \]

So the ratio is a unit of \( \mathcal{O}_k \). To get a unit \( > 1 \), we compute

\[ \frac{13 + 7 \sqrt[3]{6} + 4 \sqrt[3]{36}}{-5 + \sqrt[3]{6} + \sqrt[3]{36}} = 109 + 60 \sqrt[3]{6} + 33 \sqrt[3]{36} \]

It turns out that \( u \) is a fundamental unit of \( \mathcal{O}_k \), but [2, Lemma 2] does not apply, since for the fundamental unit \( U \), \( U^2 > (3^5 - 7)^{2/3} \approx 38.189 \), a lower bound which is too small to conclude \( U^2 > u \).

**Theorem 2.** The fundamental unit of \( \mathbb{Z}[\sqrt[3]{6}] \) is \( u = 109 + 60 \sqrt[3]{6} + 33 \sqrt[3]{36} \).

**Proof.** We follow the same approach as [3, Thm. 2], essentially just replacing 123 in [3] by 327. Write \( u = \rho^j \) with \( \rho^3 + a\rho^2 + b\rho + c, c = -1 \).

If \( u = \rho^2 \) then \( 327 = a^2 - 2b, 3 = b^2 + 2a \). Solving for \( a \) in the second equation turns the first one into

\[ b^4 - 6b^2 - 8b - 1299 = 0, \]

so \( b \mid 1299 = 3 \cdot 433 \). No divisor works.

If \( u = \rho^3 \) then \( 327 = -a^3 + 3ab + 3 \) and \( 3 = b^3 + 3ab + 3 \), and there is no solution by the same method as in [3].

If \( u = \rho^p \) for \( p \) an odd prime, then \( N_{k/\mathbb{Q}}(\rho + 1) = 2 - a + b \) is a positive integer which divides \( N_{k/\mathbb{Q}}(u + 1) = 332 = 2^2 \cdot 83 \), \( N_{k/\mathbb{Q}}(\rho - 1) = -a - b \) is a positive integer which divides \( N_{k/\mathbb{Q}}(u - 1) = 324 = 2^2 \cdot 3^4 \), and

\[ 327 \equiv -a \mod p, \ 3 \equiv b \mod p. \]

This is the same as

\[ 2 - a + b \equiv 332 \mod p, \ -a - b \equiv 324 \mod p. \]

Here is the table of values of \( 2 - a + b \) and \( -a - b \) along with the corresponding primes \( p \):

<table>
<thead>
<tr>
<th>( 2 - a + b )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>83</th>
<th>166</th>
<th>332</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>331</td>
<td>2, 3, 5, 11</td>
<td>2, 41</td>
<td>3, 83</td>
<td>2, 83</td>
<td>arb.</td>
</tr>
</tbody>
</table>
Following the same procedure as in [3], we eliminate primes $p \geq 7$ by checking the resulting cubic polynomial for a putative $\rho$ has discriminant not divisible by $\text{disc}(\mathcal{O}_K) = 2^3 3^5$ (it also must be a divisor of $\text{disc}(\mathbf{Z}[u]) = -2^4 3^{11} 7^2$, but this won’t be needed). We already eliminated $p = 2, 3$. The prime 5 appears often in the above tables, so we handle it instead by finding a residue field $\mathcal{O}_K / p \cong \mathbf{F}_p$ where $u$ is not a fifth power. Choose $p \equiv 1 \mod 5$, say $p = 11$. Since $X^3 - 6 \equiv (X + 3)(X^2 + 8X + 9) \mod 11$, there is $p$ with norm 11. Then

$$u = \rho^5 \Rightarrow u \equiv \rho^5 \mod p \equiv \pm 1.$$ 

However, neither $N_{K/\mathbf{Q}}(u + 1)$ nor $N_{K/\mathbf{Q}}(u - 1)$ is divisible by 11, so $u \neq \rho^5$. 

![The splitting field of $X^3 - 6$ over $\mathbf{Q}$](image)

**Theorem 3.** The field $k = \mathbf{Q}(\sqrt[3]{6})$ has ring of integers $\mathbf{Z}[\sqrt[3]{6}]$, class number 1, discriminant $-2^2 3^5$, and fundamental unit $u = 109 + 60 \sqrt[3]{6} + 33 \sqrt[3]{36}$. The ramified primes 2 and 3 factor as

$$2 = (2 - \sqrt[3]{6})^3 u, \quad 3 = \pi^3 v,$$

where $\pi = 3 + 2 \sqrt[3]{6} + \sqrt[3]{36}$ and $v = 1/u$. The minimal polynomial of $u$ over $\mathbf{Q}$ is $T^3 - 327T^2 + 3T - 1$ and of $\pi$ is $T^3 - 9T^2 - 9T - 3$.

We now turn to $K$. Following [2],

$$\text{disc}(K) = \text{disc}(F) \text{disc}(k)^2 = -2^4 3^{11}, \quad h(K)R(K) = (h(k)R(k))^2 = (\log u)^2.$$

The prime 3 is totally ramified: $3 \mathcal{O}_K = (\eta)^6$, where $\eta \equiv \sqrt[3]{-3}/\pi$. The (principal) prime factor of 2 in $\mathcal{O}_K$ remains prime in $\mathcal{O}_K$: $2 \mathcal{O}_K = (2 - \sqrt[3]{6})^3$.

As in [2], $\mathcal{O}_K = \mathcal{O}_k \oplus \mathcal{O}_k \theta$, where $\theta \equiv (\omega - 1)/\pi$. Since

$$\theta \overline{\theta} = \frac{3}{\pi^2} = \pi v = 3 + 2 \sqrt[3]{6} - 2 \sqrt[3]{36}, \quad \theta + \overline{\theta} = -\frac{3}{\pi} = -\pi^2 v = 3 - 3 \sqrt[3]{36},$$

the minimal polynomial of $\theta$ over $k$ is $f(T) = T^2 - (3 - \sqrt[3]{36})T + (3 \sqrt[3]{6} - 2 \sqrt[3]{36})$, so the minimal polynomial of $\theta$ over $\mathbf{Q}$ is

$$g(T) = f(\sigma(f)) \sigma^2(f) = T^6 - 97^5 + 36 T^4 - 81 T^3 + 72 T^2 + 27 T + 3.$$ 

Since $\text{disc}(g(T)) = -2^4 3^{11} 7^2 5^2$, $\mathcal{O}_K \neq \mathbf{Z}[\theta]$.

The Minkowski bound for $K$ is

$$\frac{6! (4/\pi)^3}{6^6} 2^2 3^5 \sqrt[3]{3} = \frac{960 \sqrt[3]{3}}{\pi^3} \approx 53.626.$$ 

So we want to factor all primes $\leq 53$, hopefully many will have principal ideal factors.

We already checked the ramified primes 2 and 3 have principal prime factors in $\mathcal{O}_K$, so we turn to unramified primes $p$. The only time $p$ might not have a principal prime factor in $\mathcal{O}_K$ is if $p \equiv 1 \mod 3$ and $X^3 - 6 \mod p$ has a root (hence 3 roots). For $p \leq 53$, this happens only for $p = 7, 37$:

$$X^3 - 6 \equiv (X + 1)(X + 2)(X + 4) \mod 7, \quad X^3 - 6 \equiv (X + 6)(X + 8)(X + 23) \mod 37.$$

Thus 7 and 37 split completely in $\mathcal{O}_K$. To determine if they have principal prime factors, we compute $N_{K/k}(\theta - m) = g(m)$ for various integers $m$, hoping to see a 7 or 37 arise. This would correspond to $m \mod 7$ or $m \mod 37$ being a root of $g(T)$. Since $g(-1) \equiv 0 \mod 7$, we compute $N_{K/\mathbf{Q}}(\theta + 1) = g(-1) = 5^2$. Therefore 7 factors principally. Since $g(-2) \equiv 0 \mod 37$, we compute $N_{K/\mathbf{Q}}(\theta + 2) = 7^2 37$. Thus 37 factors principally, so $h(K) = 1$. 

<table>
<thead>
<tr>
<th>$-a - b$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>17, 19</td>
<td>2, 7, 23</td>
<td>3, 107</td>
<td>2, 5</td>
<td>2, 3, 53</td>
<td>3, 5, 7</td>
<td>2, 3, 13</td>
<td>2, 3, 17</td>
</tr>
<tr>
<td>$-a - b$</td>
<td>27</td>
<td>36</td>
<td>54</td>
<td>81</td>
<td>108</td>
<td>162</td>
<td>324</td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>3, 11</td>
<td>2, 3</td>
<td>3, 5</td>
<td>3</td>
<td>2, 3</td>
<td>2, 3</td>
<td>arb.</td>
<td></td>
</tr>
</tbody>
</table>
By (2), $R(K) = (\log u)^2$, so $u$ and $\sigma u$ generate a subgroup of index 3 in the units of $O_K \pmod{\text{torsion}}$. To find a unit which together with $u$ forms a pair of fundamental units, consider

$$\delta \overset{\text{def}}{=} \frac{\sigma \eta}{\eta} = \frac{\pi}{\sigma \pi}.$$  

By exactly the same calculations as in [3], $\{u, \delta\}$ and $\{\delta, \overline{\delta}\}$ are both pairs of fundamental units.

**Theorem 4.** The field $K = \mathbb{Q} (\sqrt[3]{6}, \omega)$ has class number 1, discriminant $-2^4 3^{11}$, and regulator $(\log u)^2$, where $u = 109 + 60 \sqrt[3]{6} + 33 \sqrt[3]{36}$. The ramified primes 2 and 3 factor as

$$2 = (2 - \sqrt[3]{6})^3, \quad 3 = (\eta)^6,$$

where $\eta = \sqrt{-3}/\pi$, $\pi = 3 + 2 \sqrt[3]{6} + \sqrt[3]{36}$. The ring of integers of $K$ is

$$O_K = O_k \oplus O_k \theta,$$

where $\theta = (\omega - 1)/\pi$. The unit group of $O_K$ has six roots of unity, rank 2, and bases $\{u, \delta\}$ and $\{\delta, \overline{\delta}\}$, where $\delta = \pi/\sigma(\pi)$.

There is no power basis for $O_K$. See [1].

**References**


[3] Conrad, K. The Splitting Field of $X^3 - 5$ over $\mathbb{Q}$.