GALOIS GROUPS AS PERMUTATION GROUPS

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1. Introduction

A Galois group is a group of field automorphisms under composition. By looking at the effect of a Galois group on field generators we can interpret the Galois group as permutations, which makes it a subgroup of a symmetric group. This makes Galois groups into relatively concrete objects and is particularly effective when the Galois group turns out to be a symmetric or alternating group.

2. Automorphisms of fields as permutations of roots

The *Galois group of a polynomial* $f(T) \in K[T]$ over K is defined to be the Galois group of a splitting field for $f(T)$ over K. We do not require $f(T)$ to be irreducible in $K[T]$.

Example 2.1. The polynomial $T^4 - 2$ has splitting field $\mathbf{Q}(\sqrt[4]{2}, i)$ over $\mathbf{Q}$, so the Galois of $T^4 - 2$ over $\mathbf{Q}$ is isomorphic to D_4. The splitting field of $T^4 - 2$ over $\mathbf{R}$ is $\mathbf{C}$, so the Galois group of $T^4 - 2$ over $\mathbf{R}$ is $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \{z \mapsto z, z \mapsto \overline{z}\}$, which is cyclic of order 2.

Example 2.2. The Galois group of $(T^2-2)(T^2-3)$ over $\mathbf{Q}$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Its Galois group over $\mathbf{R}$ is trivial since the polynomial splits completely over $\mathbf{R}$.

Writing $f(T) = (T - r_1) \cdots (T - r_n)$, the splitting field of $f(T)$ over K is $K(r_1, \ldots, r_n)$. Each σ in the Galois group of $f(T)$ over K permutes the r_i's since σ fixes K and therefore $f(r) = 0 \Rightarrow f(\sigma(r)) = 0$. The automorphism σ is completely determined by its permutation of the r_i's since the r_i's generate the splitting field over K. A permutation of the r_i's can be viewed as a permutation of the subscripts $1, 2, \ldots, n$.

Example 2.3. Consider the Galois group of $T^4 - 2$ over $\mathbf{Q}$. The polynomial has 4 roots: $\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}$. Two generators of $\mathrm{Gal}(\mathbf{Q}(\sqrt[4]{2}, i)/\mathbf{Q})$ are the automorphisms r and s where $r(\sqrt[4]{2}) = i\sqrt[4]{2}$ and $r(i) = i$, and $s(\sqrt[4]{2}) = \sqrt[4]{2}$ and $s(i) = -i$. The effect of the Galois group on $\sqrt[4]{2}$ and i is in Table 1.

Automorphism	1	r	r^2	r^3	s	rs	r^2s	r^3s
Value on $\sqrt[4]{2}$	$\sqrt[4]{2}$	$i\sqrt[4]{2}$	$-\sqrt[4]{2}$	$-i\sqrt[4]{2}$	$\sqrt[4]{2}$	$i\sqrt[4]{2}$	$-\sqrt[4]{2}$	$-i\sqrt[4]{2}$
Value on i	i	i	i	i	$-i$	$-i$	$-i$	$-i$

Table 1.

The effect of r on the roots of $T^4 - 2$ is

$$r(\sqrt[4]{2}) = i\sqrt[4]{2}, \quad r(i\sqrt[4]{2}) = -\sqrt[4]{2}, \quad r(-\sqrt[4]{2}) = -i\sqrt[4]{2}, \quad r(-i\sqrt[4]{2}) = \sqrt[4]{2},$$

which is a 4-cycle, while the effect of s on the roots of $T^4 - 2$ is

$$s(\sqrt[4]{2}) = \sqrt[4]{2}, \quad s(i\sqrt[4]{2}) = -i\sqrt[4]{2}, \quad s(-i\sqrt[4]{2}) = i\sqrt[4]{2}, \quad s(-\sqrt[4]{2}) = -\sqrt[4]{2},$$

which swaps $i\sqrt[4]{2}$ and $-i\sqrt[4]{2}$ while fixing $\sqrt[4]{2}$ and $-\sqrt[4]{2}$. So s is a 2-cycle on the roots.

Indexing the roots of $T^4 - 2$ as

$$r_1 = \sqrt[4]{2}, \quad r_2 = i\sqrt[4]{2}, \quad r_3 = -\sqrt[4]{2}, \quad r_4 = -i\sqrt[4]{2}, \tag{2.1}$$

the automorphism r acts on the roots like (1234) and the automorphism s acts on the roots like (24). With this indexing of the roots, the Galois group of $T^4 - 2$ over $\mathbf{Q}$ becomes the group of permutations in S_4 in Table 2.

Automorphism	1	r	r^2	r^3	s	rs	r^2s	r^3s
Permutation	(1)	(1234)	(13)(24)	(1432)	(24)	(12)(34)	(13)	(14)(23)

TABLE 2.

Example 2.4. If we label the roots of $(T^2 - 2)(T^2 - 3)$ as

$$r_1 = \sqrt{2}, \quad r_2 = -\sqrt{2}, \quad r_3 = \sqrt{3}, \quad r_4 = -\sqrt{3},$$

then the Galois group of $(T^2 - 2)(T^2 - 3)$ over $\mathbf{Q}$ becomes the following subgroup of S_4:

$$(1), \quad (12), \quad (34), \quad (12)(34). \tag{2.2}$$

Numbering the roots of $f(T)$ in different ways can identify the Galois group with different subgroups of S_n.

Example 2.5. Renaming $\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}$ *in this order* as r_2, r_4, r_3, r_1 identifies the Galois group of $T^4 - 2$ over $\mathbf{Q}$ with the subgroup of S_4 in Table 3, which is not the same subgroup of S_4 in Table 2.

Automorphism	1	r	r^2	r^3	s	rs	r^2s	r^3s
Permutation	(1)	(1243)	(14)(23)	(1342)	(14)	(13)(24)	(23)	(12)(34)

TABLE 3.

Example 2.6. If we label $\sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}$ in this order as r_2, r_4, r_1, r_3 then the Galois group of $(T^2 - 2)(T^2 - 3)$ over $\mathbf{Q}$ turns into the following subgroup of S_4:

$$(1), \quad (13), \quad (24), \quad (13)(24). \tag{2.3}$$

This is not the same subgroup as (2.2).

In general, associating to each σ in the Galois group of $f(T)$ over K its permutation on the roots of $f(T)$, viewed as a permutation of the subscripts of the roots when we list them as $r_1, \ldots, r_n$, is a homomorphism from the Galois group to S_n. This homomorphism is injective since its kernel is trivial: an element of the Galois group that fixes each r_i is the identity on the splitting field. Thinking about the Galois group of a polynomial with degree n as a subgroup of S_n is the original viewpoint of Galois. (The description of Galois theory in terms of field automorphisms is due to Dedekind and, with more abstraction, Artin.)

Two different choices for indexing the roots of $f(T)$ can lead to different subgroups of S_n, but they will be conjugate subgroups. For instance, the subgroups in Tables 2 and 3 are conjugate by the permutation $\left(\begin{smallmatrix}1234\\2431\end{smallmatrix}\right) = (124)$, which is the permutation turning one indexing of the roots into the other, and the subgroups (2.2) and (2.3) are conjugate by

$\binom{1234}{2413} = (1243)$. Although the Galois group of $f(T)$ over K does not have a canonical embedding into S_n in general, its image in S_n is well-defined *up to an overall conjugation*. For example, without fixing an indexing of the roots, it doesn't make sense to ask if a particular permutation like (132) is in the Galois group as a subgroup of S_n, but it does make sense to ask if the Galois group contains a permutation with a particular cycle type (like a 3-cycle).

We can speak about Galois groups of irreducible or reducible polynomials, like $T^4 - 2$ or $(T^2 - 2)(T^3 - 2)$ over $\mathbf{Q}$. Only for an irreducible polynomial does the Galois group have a special property, called transitivity, when we turn the Galois group into a subgroup of S_n. A subgroup $G \subset S_n$ is called *transitive* when, for all $i \neq j$ in $\{1, 2, \ldots, n\}$, there is a permutation in G sending i to j.

Example 2.7. The subgroups of S_4 in Tables 2 and 3 are transitive. This corresponds to the fact that for each pair of roots of $T^4 - 2$ there is an element of its Galois group over $\mathbf{Q}$ taking the first root to the second.

Example 2.8. The subgroup of S_4 in (2.2) is not transitive since no element of the subgroup takes 1 to 3. This corresponds to the fact that an element of $\mathrm{Gal}(\mathbf{Q}(\sqrt{2}, \sqrt{3})/\mathbf{Q})$ can't send $\sqrt{2}$ to $\sqrt{3}$.

Being transitive is not a property of an abstract group. It is a property of subgroups of S_n.[1] A conjugate subgroup of a transitive subgroup of S_n is also transitive since conjugation on S_n amounts to listing the numbers from 1 to n in a different order.

Theorem 2.9. *Let $f(T) \in K[T]$ be a separable polynomial of degree n.*

(a) *If $f(T)$ is irreducible in $K[T]$ then its Galois group over K has order divisible by n.*
(b) *The polynomial $f(T)$ is irreducible in $K[T]$ if and only if its Galois group over K is a transitive subgroup of S_n.*

Proof. (a) For a root r of $f(T)$ in K, $[K(r) : K] = n$ is a factor of the degree of the splitting field over K, which is the size of the Galois group over K.

(b) First suppose $f(T)$ is irreducible. For two roots r_i and r_j of $f(T)$, we can write $r_j = \sigma(r_i)$ for some σ in the Galois group of $f(T)$ over K. Therefore the Galois group, as a subgroup of S_n, sends i to j, so it is a transitive subgroup. Now suppose $f(T)$ is reducible (so $n \geq 2$). It is a product of distinct irreducibles since it is separable. Let r_i and r_j be roots of different irreducible factors of $f(T)$. These irreducible factors are the minimal polynomials of r_i and r_j over K. For each σ in the Galois group of $f(T)$ over K, $\sigma(r_i)$ has the same minimal polynomial over K as r_i, so we can't have $\sigma(r_i) = r_j$. Therefore, as a subgroup of S_n, the Galois group of $f(T)$ does not send i to j, so it is not a transitive subgroup of S_n. □

3. The group S_p as a Galois group

We will give a criterion that implies a Galois group of prime degree p is as large as possible, namely S_p.

Lemma 3.1. *In S_p, a permutation of order p is a p-cycle.*

[1] More generally, it is a property of groups equipped with a specific action on a set. Subgroups of S_n have a natural action on the set $\{1, 2, \ldots, n\}$.

Proof. Let $\pi \in S_p$ have order p and decompose into disjoint nontrivial cycles as $\pi = \pi_1\pi_2\cdots\pi_r$, with n_i being the order of π_i.

Method 1: Suppose π is not a p-cycle. Since we're in S_p, each π_i is therefore a cycle of length less than p, so each n_i is less than p (in fact the length of the cycle π_i is n_i). Since the order of a product of disjoint cycles is the least common multiple of the orders of the cycles, $p = \mathrm{lcm}(n_1, \ldots, n_r)$. However, each n_i is less than p, so not divisible by p, and therefore $\mathrm{lcm}(n_1, \ldots, n_r)$ is not divisible by p. This is a contradiction.

Method 2: the order of $\pi_1\pi_2\cdots\pi_r$ is $\mathrm{lcm}(n_1, \ldots, n_r)$. Therefore $p = \mathrm{lcm}(n_1, \ldots, n_r)$. Since p is a prime number and $n_i > 1$, we have $n_i = p$ for all i. Thus π is a product of disjoint p-cycles. Since π is in S_p, there can't be even two disjoint p-cycles, so π is a single p-cycle. □

Theorem 3.2. *Let $f(T) \in \mathbf{Q}[T]$ be an irreducible polynomial of prime degree p with all but two roots in $\mathbf{R}$. The Galois group of $f(T)$ over $\mathbf{Q}$ is isomorphic to S_p.*

Proof. Let $L = \mathbf{Q}(r_1, \ldots, r_p)$ be the splitting field of $f(T)$ over $\mathbf{Q}$. The permutations of the r_i's by $\mathrm{Gal}(L/\mathbf{Q})$ provide an embedding $\mathrm{Gal}(L/\mathbf{Q}) \hookrightarrow S_p$ and $|\mathrm{Gal}(L/\mathbf{Q})|$ is divisible by p by Theorem 2.9, so $\mathrm{Gal}(L/\mathbf{Q})$ contains an element of order p by Cauchy's theorem. In S_p, the only permutations of order p are p-cycles (Lemma 3.1). So the image of $\mathrm{Gal}(L/\mathbf{Q})$ in S_p contains a p-cycle.

We may take L to be a subfield of $\mathbf{C}$, since $\mathbf{C}$ is algebraically closed. Complex conjugation restricted to L is a member of $\mathrm{Gal}(L/\mathbf{Q})$. Since $f(T)$ has only two non-real roots by hypothesis, complex conjugation transposes two of the roots of $f(T)$ and fixes the others. Therefore $\mathrm{Gal}(L/\mathbf{Q})$ contains a transposition of the roots of $f(T)$. (This is the reason for the hypothesis about all but two roots being real.)

We now show the only subgroup of S_p containing a p-cycle and a transposition is S_p, so $\mathrm{Gal}(L/\mathbf{Q}) \cong S_p$. By suitable labeling of the numbers from 1 to p, we may let 1 be a number moved by the transposition, so our subgroup contains a transposition $\tau = (1a)$. Let σ be a p-cycle in the subgroup. As a p-cycle, σ acts on $\{1, 2, \ldots, p\}$ by a single orbit, so some σ^i with $1 \leq i \leq p-1$ sends 1 to a: $\sigma^i = (1a\ldots)$. This is also a p-cycle, because σ^i has order p in S_p and all elements of order p in S_p are p-cycles, so writing σ^i as σ and suitably reordering the numbers $2, \ldots, p$ (which replaces our subgroup by a conjugate subgroup), we may suppose our subgroup of S_p contains the particular transposition (12) and the particular p-cycle $(12\ldots p)$. For $n \geq 2$, it is a theorem in group theory that the particular transposition (12) and n-cycle $(12\ldots n)$ generate S_n, so our subgroup is S_p. □

Remark 3.3. While S_p is generated by each transposition and p-cycle for p prime, it is not true that S_n is generated by eachtransposition and n-cycle for general n. For example, (13) and (1234) generate a proper subgroup of S_4 (one of the subgroups of order 8).

Example 3.4. The polynomial $T^3 - T - 1$ is irreducible in $\mathbf{Q}[T]$ since it is irreducible mod 2 or since it is a cubic without a rational root. It has one real root (approximately 1.3247), and one root of a cubic is all but two roots, so its Galois group over $\mathbf{Q}$ is isomorphic to S_3.

Example 3.5. The polynomials $T^3 - 3T - 1$ and $T^3 - 4T - 1$ are both irreducible in $\mathbf{Q}[T]$ since they are cubics without a rational root. Each polynomial has three real roots (check!), so we can't use Theorem 3.2 to determine their Galois groups over $\mathbf{Q}$.

Example 3.6. The quintic polynomial $T^5 - T - 1$ is irreducible in $\mathbf{Q}[T]$ since it is irreducible mod 3. It has one root in $\mathbf{R}$, not all but two roots in $\mathbf{R}$, so Theorem 3.2 does not tell us the Galois group.

Example 3.7. The quintic polynomial T^5-4T-1 is irreducible in $\mathbf{Q}[T]$ since it is irreducible mod 3. It has three real roots, which is all but two roots, so its Galois group over $\mathbf{Q}$ is isomorphic to S_5.

4. Alternating groups and the discriminant

The next thing we will do with Galois groups as subgroups of S_n is determine when they lie in A_n. Without fixing a labeling of the roots of $f(T)$, its Galois group is determined as a subgroup of S_n only up to conjugation, but it is still meaningful to ask if the Galois group is a subgroup of A_n since $A_n \lhd S_n$. We will introduce a numerical invariant of polynomials, called the discriminant, to determine when the Galois group is in A_n.

Definition 4.1. For a nonconstant $f(T) \in K[T]$ of degree n that factors over a splitting field as

$$f(T) = c(T - r_1)\cdots(T - r_n),$$

the *discriminant* of $f(T)$ is defined to be

$$\operatorname{disc} f = \prod_{i<j}(r_j - r_i)^2.$$

Example 4.2. The polynomial $(T-1)(T-3)(T-7)$ has discriminant $2^2 \cdot 6^2 \cdot 4^2 = 2304$.

Example 4.3. The discriminant of $T^2 + aT + b = (T - r)(T - r')$ is

$$(r - r')^2 = r^2 - 2rr' + r'^2 = (r + r')^2 - 4rr' = a^2 - 4b,$$

which is the usual discriminant of a monic quadratic polynomial.

The number $\operatorname{disc} f$ is nonzero if $f(T)$ is separable and is 0 if $f(T)$ is not separable. When $f(T)$ is separable, $\operatorname{disc} f$ is a symmetric polynomial in the r_i's, so it is fixed by $\mathrm{Gal}(K(r_1, \ldots, r_n)/K)$ and therefore $\operatorname{disc} f \in K$ by Galois theory. We have $\operatorname{disc} f \in K$ if $f(T)$ is not separable too, since in that case $\operatorname{disc} f$ is 0.

Because $\operatorname{disc} f$ is a symmetric polynomial in the roots of $f(T)$, when $f(T)$ is monic its discriminant is a polynomial in the coefficients of $f(T)$ (which are, up to sign, the elementary symmetric functions of the roots). In low-degree cases, explicit formulas for discriminants of some trinomials are

$$\begin{aligned}
\operatorname{disc}(T^2 + aT + b) &= a^2 - 4b,\\
\operatorname{disc}(T^3 + aT + b) &= -4a^3 - 27b^2,\\
\operatorname{disc}(T^4 + aT + b) &= -27a^4 + 256b^3,\\
\operatorname{disc}(T^5 + aT + b) &= 256a^5 + 3125b^4.
\end{aligned}$$

Example 4.4. The discriminant of $T^3 - T - 1$ is -23, the discriminant of $T^3 - 3T - 1$ is 81, and the discriminant of $T^3 - 4T - 1$ is 229.

More generally [3, p. 41],

$$\operatorname{disc}(T^n + aT + b) = (-1)^{n(n-1)/2}((-1)^{n-1}(n-1)^{n-1}a^n + n^n b^{n-1}),$$

and even more generally, for $0 < m < n$ and $(m, n) = 1$,

$$\operatorname{disc}(T^n + aT^m + b) = (-1)^{n(n-1)/2}b^{m-1}((-1)^{n-1}m^m(n-m)^{n-m}a^n + n^n b^{n-m}).$$

If (m, n) is not necessarily 1 then [4, Theorem 2]

$$\operatorname{disc}(T^n + aT^m + b) = (-1)^{n(n-1)/2}b^{m-1}((-1)^{n/d-1}m^{m/d}(n-m)^{(n-m)/d}a^{n/d} + n^{n/d}b^{(n-m)/d})^d,$$

where $d = (m, n)$. We will not derive these formulas here.

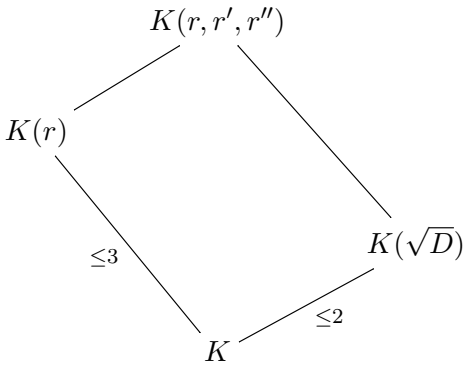
Example 4.5. Taking $m = 2$ and $n = 4$,

$$\operatorname{disc}(T^4 + aT^2 + b) = 16b(a^2 - 4b)^2.$$

Theorem 4.6. *Let K not have characteristic 2 and let $f(T)$ be a separable cubic in $K[T]$ with a root r and discriminant D. The splitting field of $f(T)$ over K is $K(r, \sqrt{D})$.*

Note we are *not* assuming $f(T)$ is irreducible here.

Proof. The roots and the discriminant of f don't change if we multiply f by a nonzero constant, so we may assume $f(T)$ is monic. Let $f(T)$ have roots r, r', and r'', so the splitting field of $f(T)$ over K is $K(r, r', r'')$.



Over $K(r)$, we can remove a linear factor and write $f(T) = (T - r)g(T)$, where $g(r) \neq 0$. Explicitly, $g(T) = (T - r')(T - r'')$. (Since f is monic, so is g.) By the quadratic formula on $g(T)$, $K(r, r', r'') = K(r, \sqrt{\operatorname{disc} g})$. It is simple to check, since f is monic, that $\operatorname{disc} f = g(r)^2 \operatorname{disc} g$, so $K(r, \sqrt{\operatorname{disc} g}) = K(r, \sqrt{\operatorname{disc} f}) = K(r, \sqrt{D})$. □

Theorem 4.7. *Let $f(T) \in K[T]$ be a separable polynomial of degree n. If K does not have characteristic 2, the embedding of the Galois group of $f(T)$ over K into S_n as permutations of the roots of $f(T)$ has image in A_n if and only if* $\operatorname{disc} f$ *is a square in K.*

Proof. Set $\delta = \prod_{i<j}(r_j - r_i) \neq 0$, so $\delta \in K(r_1, \ldots, r_n)$ and $\delta^2 = \operatorname{disc} f \in K$. Therefore $\operatorname{disc} f$ is a square in K if and only if $\delta \in K$.

For $\sigma \in \operatorname{Gal}(K(r_1, \ldots, r_n)/K)$, let $\varepsilon_\sigma = \pm 1$ be its sign as a permutation of the r_i's. By one of the definitions of the sign of a permutation,

$$\sigma(\delta) = \prod_{i<j}(\sigma(r_j) - \sigma(r_i)) = \varepsilon_\sigma \prod_{i<j}(r_j - r_i) = \varepsilon_\sigma \delta,$$

so $\sigma(\delta) = \pm\delta$. Since $\delta \neq 0$ and K doesn't have characteristic 2, $\delta \neq -\delta$. We have $\sigma \in A_n$ if and only if $\varepsilon_\sigma = 1$, so $\sigma \in A_n$ if and only if $\sigma(\delta) = \delta$. Therefore the Galois group of $f(T)$ over K is in A_n if and only if δ is fixed by the Galois group, which is the same as $\delta \in K$. □

Remark 4.8. Theorem 4.7 is completely false in characteristic 2: discriminants of polynomials are always squares in characteristic 2 but S_n can occur as a Galois group. For example, if F is a field of characteristic 2, then over $F(u)$ the polynomial $T^3 + uT + u$ is separable and irreducible with discriminant u^2 (a square) and Galois group S_3.

Theorem 4.7 lets us determine the Galois groups of irreducible cubic polynomials outside of characteristic 2.

Theorem 4.9. *Let K not have characteristic 2 and let $f(T)$ be a separable irreducible cubic in $K[T]$.*

(a) *If* $\operatorname{disc} f$ *is a square in K then the Galois group of $f(T)$ over K is isomorphic to A_3.*
(b) *If* $\operatorname{disc} f$ *is not a square in K then the Galois group of $f(T)$ over K is isomorphic to S_3.*

Proof. Since $K(r)$ is inside the splitting field, the Galois group is divisible by $[K(r) : K] = 3$. The permutations of the roots of $f(T)$ by its Galois group over K gives an embedding of the Galois group into S_3, so the image is either A_3 or S_3 since these are the only subgroups of S_3 with size divisible by 3. Theorem 4.7 says the image is in A_3 (and thus equal to A_3) if and only if $\operatorname{disc} f$ is a square in K.

This can also be proved using the formula for the splitting field of a cubic in Theorem 4.6. □

Example 4.10. In Example 3.4 we saw $T^3 - T - 1$ has Galois group S_3 over $\mathbf{Q}$. We can see again that $T^3 - T - 1$ has Galois group S_3 over $\mathbf{Q}$ since its discriminant is -23 (Example 4.4) and this is not a rational square.

Example 4.11. The Galois groups of $T^3 - 3T - 1$ and $T^3 - 4T - 1$ over $\mathbf{Q}$ were left undetermined in Example 3.5 since all of their roots are real. Now we can compute the Galois groups. From Example 4.4, $T^3 - 3T - 1$ has discriminant 81 (a square) and $T^3 - 4T - 1$ has discriminant 229 (a prime). Therefore $T^3 - 3T - 1$ has Galois group A_3 over $\mathbf{Q}$ and $T^3 - 4T - 1$ has Galois group S_3 over $\mathbf{Q}$. So although $T^3 - 3T - 1$ and $T^3 - 4T - 1$ both have all real roots, their Galois groups are not isomorphic. Each root of $T^3 - 3T - 1$ generates the splitting field over $\mathbf{Q}$, but this is not true for a root of $T^3 - 4T - 1$.

Remark 4.12. The cubics $T^3 - 2T + 1$ and $T^3 - 7T - 6$ have respective discriminants 5 and $400 = 20^2$, but this does *not* mean their Galois groups over $\mathbf{Q}$ are S_3 and A_3. Both polynomials are reducible (factoring as $(T-1)(T^2+T-1)$ and $(T+1)(T+2)(T-3)$). Do not forget to check that a cubic is irreducible before you use Theorem 4.9!

The following fantastic theorem of Dedekind uses factorization of a polynomial mod p to tell us when a Galois group over $\mathbf{Q}$ contains permutations with particular cycle structure.

Theorem 4.13 (Dedekind)**.** *Let $f(T) \in \mathbf{Z}[T]$ be monic irreducible over $\mathbf{Q}$ of degree n. For a prime p not dividing* $\operatorname{disc} f$*, let the monic irreducible factorization of $f(T)$* mod p *be*

$$f(T) \equiv \pi_1(T) \cdots \pi_k(T) \bmod p$$

and set $d_i = \deg \pi_i(T)$, so $d_1 + \cdots + d_k = n$. The Galois group of $f(T)$ over $\mathbf{Q}$, viewed as a subgroup of S_n, contains a permutation of type $(d_1, \ldots, d_k)$.

The nicest proof of Theorem 4.13 uses algebraic number theory and is beyond the scope of these notes. More elementary proofs are in [1, pp. 398–400] and [2, pp. 302–304].

Example 4.14. We compute the Galois group of $T^4 - T - 1$ over $\mathbf{Q}$ using Theorem 4.13.

This polynomial is irreducible mod 2, so it is irreducible over $\mathbf{Q}$. Let its roots be r_1, r_2, r_3, r_4. The extension $\mathbf{Q}(r_1)/\mathbf{Q}$ has degree 4, so the Galois group of $T^4 - T - 1$ over $\mathbf{Q}$ has order divisible by 4. Since the Galois group embeds into S_4, its size is either 4, 8, 12, or 24. The discriminant of $T^4 - T - 1$ is -283, which is not a rational square, so the Galois group is not a subgroup of A_4. This eliminates the possibility of the Galois group having order 12, because the only subgroup of S_4 with order 12 is A_4. (Quite generally, the

only subgroup of index 2 in S_n is A_n for $n \geq 2$.) There are subgroups of S_4 with orders 4, 8, and (of course) 24 outside of A_4, so no other size but 12 is eliminated yet.

Using Theorem 4.13 with $p = 7$, the irreducible factorization of $T^4 - T - 1$ mod 7 is

$$T^4 - T - 1 \equiv (T + 4)(T^3 + 3T^2 + 2T + 5) \bmod 7.$$

Theorem 4.13 says the Galois group of $T^4 - T - 1$ over $\mathbf{Q}$ contains a permutation of the roots with cycle type $(1, 3)$, so the group has order divisible by 3. This proves it is S_4.

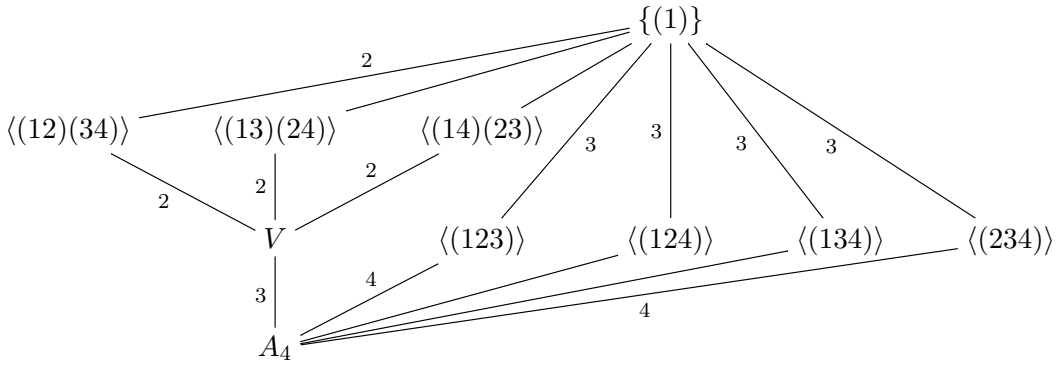
Example 4.15. We find the Galois group of $T^4+8T+12$ over $\mathbf{Q}$. This is reducible mod p for all small p, so the reduction mod p test doesn't help us prove the polynomial is irreducible over $\mathbf{Q}$. (In fact, the polynomial factors mod p for all p, so the reduction mod p test never works.) Let's look at *how* the polynomial factors into irreducibles modulo different primes:

$$\begin{aligned} T^4 + 8T + 12 &\equiv (T + 1)(T^3 + 4T^2 + T + 2) \bmod 5, \\ T^4 + 8T + 12 &\equiv (T^2 + 4T + 7)(T^2 + 13T + 9) \bmod 17. \end{aligned}$$

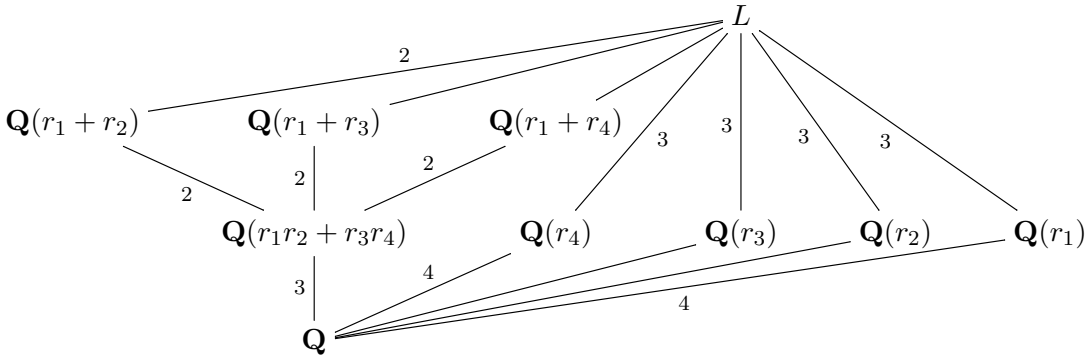
These are only consistent with $T^4 + 8T + 12$ being irreducible over $\mathbf{Q}$ (why?).

Since $T^4 + 8T + 12$ is irreducible over $\mathbf{Q}$, its Galois group over $\mathbf{Q}$ has size divisible by 4. Its discriminant is $331776 = 576^2$, a rational square, so the Galois group is a subgroup of A_4 and therefore has size 4 or 12. From the factorization of the polynomial mod 5 above, the Galois group contains a permutation of the roots whose cycle type is $(1, 3)$, which is a 3-cycle, so the Galois group has order divisible by 3, and thus its size is 12. So the Galois group of $T^4 + 8T + 12$ over $\mathbf{Q}$ is isomorphic to A_4: the even permutations of the roots extend to automorphisms of the splitting field over $\mathbf{Q}$, while the odd permutations do not.

Let's list all the subfields of the splitting field of $T^4 + 8T + 12$ over $\mathbf{Q}$. Here is the lattice (upside down) of subgroups of A_4.



The corresponding subfield lattice of $L = \mathbf{Q}(r_1, r_2, r_3, r_4)$ is as follows.



The normal subgroups of A_4 are $\{1\}$, V, and A_4, so the only subfield of L that is Galois over $\mathbf{Q}$ other than L and $\mathbf{Q}$ is $\mathbf{Q}(r_1r_2+r_3r_4)$. Since $[L:\mathbf{Q}(r_1)]=3$ is prime and $r_2 \notin \mathbf{Q}(r_1)$, we have $L=\mathbf{Q}(r_1,r_2)$, so $[\mathbf{Q}(r_1,r_2):\mathbf{Q}]=12$.

The sums r_1+r_2, r_1+r_3, and r_1+r_4 are roots of T^6-48T^2-64 (check!) and $r_1r_2+r_3r_4$ is a root of $T^3-48T-64$. Roots of $T^3-48T-64$ are squares of roots of T^6-48T^2-64. It is left to the reader to check that $r_1r_2+r_3r_4=(r_1+r_2)^2=(r_3+r_4)^2$.

Remark 4.16. Our Galois group computations have an application to "nonexistent fields." If r is a root of $T^4+8T+12$ then $[\mathbf{Q}(r):\mathbf{Q}]=4$ and there is *no* quadratic field in $\mathbf{Q}(r)$: such a field would, by the Galois correspondence, lead to a subgroup of index 2 in A_4 and there is no such subgroup. More generally, for $k \geq 2$ there are complex numbers that have degree 2^k over $\mathbf{Q}$ and the field they generate over $\mathbf{Q}$ contains no quadratic extension of $\mathbf{Q}$.

Example 4.17. Let's determine the Galois group of T^5-T-1 over $\mathbf{Q}$, which was left unresolved in Example 3.6. Its irreducible factorization mod 2 is

$$T^5-T-1=(T^2+T+1)(T^3+T^2+1) \bmod 2.$$

Because the polynomial is irreducible over $\mathbf{Q}$, 5 divides the size of the Galois group. From the mod 2 factorization, the Galois group contains a permutation of the roots with cycle type $(2,3)$, which has order 6, so the Galois group has size divisible by $5\cdot 6=30$. Since the Galois group is a subgroup of S_5, its size is either 30, 60, or 120.

It turns out that there is no subgroup of S_5 with order 30 and the only subgroup of order 60 is A_5. The discriminant of $f(x)$ is $2869=19\cdot 151$, which is not a rational square, so the Galois group is not in A_5 by Theorem 4.7. Therefore the Galois group is S_5.

Theorem 3.2 gave us a sufficient condition for an irreducible in $\mathbf{Q}[T]$ of prime degree p to have Galois group S_p: all but 2 roots are real. This condition does not apply to T^3-4T-1 or T^5-T-1, although by Examples 4.11 and 4.17 their Galois groups over $\mathbf{Q}$ are S_3 and S_5. Using Theorem 4.13, there is a different "all but 2 roots" hypothesis that implies a Galois group is S_p.

Corollary 4.18. *Let $f(T) \in \mathbf{Z}[T]$ be monic irreducible over $\mathbf{Q}$ of prime degree p. If there is a prime number ℓ not dividing $\operatorname{disc} f$ such that $f(T) \bmod \ell$ has all but two roots in $\mathbf{F}_\ell$, then the Galois group of $f(T)$ over $\mathbf{Q}$ is isomorphic to S_p.*

Proof. The proof of Theorem 3.2 can be used again except for the step explaining why the Galois group of $f(T)$ over $\mathbf{Q}$ contains a transposition. In Theorem 3.2 this came from the use of complex conjugation to transpose two non-real roots, assuming there are only two non-real roots. We aren't assuming that now. By hypothesis the factorization of $f(T)$ mod ℓ has all linear factors except for one quadratic irreducible factor. Therefore Theorem 4.13 says the Galois group contains a permutation of the roots with cycle type $(1,1,\ldots,1,2)$, which is a transposition in S_p. □

Example 4.19. From Examples 3.4 and 4.11, the polynomials T^3-T-1 with discriminant -23 and T^3-4T-1 with discriminant 229 each have Galois group S_3 over $\mathbf{Q}$. We can check this again with Corollary 4.18: $T^3-T-1 \bmod 5$ has one root in $\mathbf{F}_5$ and $T^3-4T-1 \bmod 2$ has one root in $\mathbf{F}_2$.

Example 4.20. In Example 3.7 we saw T^5-4T-1 has Galois group S_5 over $\mathbf{Q}$ because it has all but two real roots. We can also compute the Galois group with Corollary 4.18: the discriminant is -259019 (a negative prime number) and $T^5-4T-1 \bmod 19$ has all but two roots in $\mathbf{F}_{19}$.

Example 4.21. Returning to Example 4.17, we can show the Galois group over $\mathbf{Q}$ of $T^5 - T - 1$ is S_5 in another way: $T^5 - T - 1$ has discriminant $2869 = 19 \cdot 151$ and by a computer search $T^5 - T - 1 \bmod 163$ has all but two roots in $\mathbf{F}_{163}$.

Remark 4.22. We can't use Corollary 4.18 to show $T^4 - T - 1$ has Galois group S_4 over $\mathbf{Q}$ (Example 4.14) since 4 is not prime.

If we seek an analogue of Theorem 3.2 for a Galois group to be isomorphic to A_p, using 3-cycles in place of transpositions, there is no analogue since an irreducible polynomial over $\mathbf{Q}$ can't have all but three roots in $\mathbf{R}$ (the number of non-real roots is always even). But $f(T) \bmod \ell$ could have all but three roots in $\mathbf{F}_\ell$ for some ℓ! This suggests the next result.

Corollary 4.23. *Let $f(T) \in \mathbf{Z}[T]$ be monic irreducible over $\mathbf{Q}$ of prime degree $p \geq 3$ with* disc f *a perfect square. If there is a prime number ℓ not dividing* disc f *such that $f(T) \bmod \ell$ has all but three roots in $\mathbf{F}_\ell$, then the Galois group of $f(T)$ over $\mathbf{Q}$ is isomorphic to A_p.*

Proof. Let G be the Galois group, so G is a subgroup of A_p since disc f is a square. The Galois group has order divisible by p, so it contains a p-cycle. From the factorization of $f(T) \bmod \ell$ and Theorem 4.13, G contains a 3-cycle. It is a theorem of C. Jordan that for every prime $p \geq 3$, each p-cycle and 3-cycle in S_p generate A_p, so $G \cong A_p$. □

Example 4.24. The polynomial $T^5 + 20T + 16$ has discriminant $2^{16}5^6$. It is irreducible mod 3, so it's irreducible over $\mathbf{Q}$. Modulo 7, its irreducible factorization is

$$T^5 + 20T + 16 \equiv (T - 4)(T - 5)(T^3 + 2T^2 + 5T + 5) \bmod 7.$$

This has all but three roots in $\mathbf{F}_7$, so the Galois group of $T^5 + 20T + 16$ over $\mathbf{Q}$ is isomorphic to A_5.

In Table 4 are the trinomial polynomials whose Galois group over $\mathbf{Q}$ has been computed by the methods of this section.

$f(T)$	Galois group over $\mathbf{Q}$
$T^3 - T - 1$	S_3
$T^3 - 3T - 1$	A_3
$T^3 - 4T - 1$	S_3
$T^4 - T - 1$	S_4
$T^4 + 8T + 12$	A_4
$T^5 - T - 1$	S_5
$T^5 - 4T - 1$	S_5
$T^5 + 20T + 16$	A_5

TABLE 4.

The sufficient conditions for $f(T)$ to have Galois group S_p in Corollary 4.18 and A_p in Corollary 4.23 are also necessary, by the following hard theorem of Chebotarev that serves as a converse to Dedekind's Theorem 4.13.

Theorem 4.25 (Chebotarev)**.** *Let $f(T) \in \mathbf{Z}[T]$ be monic irreducible over $\mathbf{Q}$ of degree n. If an automorphism in the Galois group of $f(T)$ over $\mathbf{Q}$ permutes the roots of $f(T)$ with cycle type $(d_1, \ldots, d_k)$, then there are infinitely many primes ℓ not dividing* disc f *such that the monic irreducible factorization of $f(T) \bmod \ell$ is*

$$f(T) \equiv \pi_1(T) \cdots \pi_k(T) \bmod \ell$$

where the $\pi_i(T)$'s are distinct and $d_i = \deg \pi_i(T)$.

Chebotarev's theorem says more (the set of such ℓ has a positive density related to conjugacy classes in the Galois group), but the version we give above is enough for us here.

Corollary 4.26. *If $f(T) \in \mathbf{Z}[T]$ is monic irreducible of prime degree p with Galois group over $\mathbf{Q}$ isomorphic to S_p (resp., A_p) then there are infinitely many primes ℓ not dividing* disc f *such that $f(T)$* mod ℓ *has all but two roots (resp., all but three roots) in $\mathbf{F}_\ell$.*

Proof. The group S_p has a 2-cycle and the group A_p for $p > 2$ has a 3-cycle. Therefore Theorem 4.25 implies there are infinitely many primes ℓ not dividing disc f such that $f(T)$ mod ℓ is an irreducible quadratic times linear polynomials in the first case and is an irreducible cubic times linear polynomials in the second case. □

For example, $T^5 - T - 1$ mod ℓ has all but two roots in $\mathbf{F}_\ell$ when $\ell = 163, 193, 227, 307, \ldots$ (there is no simple pattern to this list, but it continues indefinitely), $T^5 - 4T - 1$ mod ℓ has all but two roots in $\mathbf{F}_\ell$ when $\ell = 19, 23, 83, 97, \ldots$, and $T^5 + 20T + 16$ has all but three roots in $\mathbf{F}_\ell$ when $\ell = 7, 11, 17, 23, \ldots$. In practice, it is easy to prove when a monic irreducible in $\mathbf{Z}[T]$ with prime degree p has Galois group A_p or S_p over $\mathbf{Q}$ by searching for a prime ℓ such that Corollary 4.18 or 4.23 applies. There are extensions of these ideas to polynomials of nonprime degree, so verifying an irreducible polynomial of degree n has Galois group over $\mathbf{Q}$ that is isomorphic to S_n or A_n in practice is easy.

Proving a Galois group is small (not S_n or A_n) is a completely separate matter, which we will not discuss here.

Corollary 4.27. *Let $f(T) \in \mathbf{Z}[T]$ be monic irreducible over $\mathbf{Q}$. The Galois group of $f(T)$ over $\mathbf{Q}$ has an element of prime order p if and only if there is a prime ℓ not dividing* disc f *such that the irreducible factors of $f(T)$* mod ℓ *have degree 1 and p. If there is one such ℓ then there are infinitely many such ℓ.*

Proof. A permutation has prime order p if and only if its cycle type is $(1, \ldots, 1, p, \ldots, p)$. Theorem 4.13 says that if there is a prime ℓ not dividing disc f such that all irreducible factors of $f(T)$ mod ℓ have degree 1 or p then there is a permutation in the Galois group of $f(T)$ over $\mathbf{Q}$ whose cycles have length 1 and p, so it has order p. Conversely, if a permutation of order p occurs in the Galois group of $f(T)$ over $\mathbf{Q}$ then Theorem 4.25 says that there are infinitely many primes ℓ not dividing disc f such that all the irreducible factors of $f(T)$ mod ℓ have degree 1 or p. □

Corollary 4.27 suggests a way to find prime factors of the order of the Galois group G of $f(T)$ over $\mathbf{Q}$: factor $f(T)$ mod ℓ for lots of primes ℓ and see which factorizations have degrees $1, \ldots, 1, p, \ldots, p$ for a prime p. All such p divide $|G|$, and all prime factors of $|G|$ arise in this way. If we want to use this idea to find all prime factors of $|G|$ then we want an effective Chebotarev theorem: a bound B such that all degree-types occur modulo the primes $\ell \leq B$. You can find such results by googling "effective chebotarev," but unfortunately the versions I have seen describe B in terms of the splitting field, *e.g.*, using $|G|$, and this isn't good if we don't yet know $|G|$.

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