CONSTRUCTING ALGEBRAIC CLOSURES

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Let K be a field. We want to construct an algebraic closure of K, *i.e.*, an algebraic extension of K which is algebraically closed. It will be built as the quotient of a polynomial ring in a very large number of variables.

For each nonconstant monic polynomial $f(X)$ in $K[X]$, let its degree be n_f and let $t_{f,1}, \ldots, t_{f,n_f}$ be independent variables. Let $A = K[\{t_{f,i}\}]$ be the polynomial ring generated over K by independent variables doubly indexed by every nonconstant monic $f \in K[X]$ and $1 \leq i \leq n_f$. This is a *very large* polynomial ring containing K.

Since $K \subset A$, we can view all $f(X)$ in $A[X]$. Let's make them all split in some $(A/I)[X]$. Let I be the ideal in A generated by the coefficients of powers of X in all the differences

$$f(X) - \prod_{i=1}^{n_f}(X - t_{f,i}) \in A[X]$$

as f runs over nonconstant monic polynomials in $K[X]$. These differences all lie in $I[X]$, so $f(X) \equiv \prod_i (X - t_{f,i})$ in $(A/I)[X]$.

We want to use a maximal ideal in place of I since working modulo a maximal ideal would give a complete splitting of all monic $f(X) \in K[X]$ over a *field*.

Lemma 1. *The ideal I is proper: $1 \notin I$.*

Proof. We will argue by contradiction and the main idea is to use the existence of a splitting field for finitely many polynomials in $K[X]$.

Suppose $1 \in I$. Then we can write 1 as a *finite* sum $\sum_{j=1}^{m} a_j c_j$, where $a_j \in A$ and $c_j \in I$: each c_j is a coefficient of some power of X in some difference

$$f_j(X) - \prod_{i=1}^{n_j}(X - t_{f_j,i}), \tag{1}$$

where n_j means $n_{f_j} = \deg f_j$. Since $f_j(X)$ is monic in $K[X]$ and $n_j = \deg(f_j)$. there is a (finite) field extension L/K over which the polynomials $f_1(X), \ldots, f_m(X)$ all split completely, say $f_j(X) = \prod_{i=1}^{n_j}(X - r_{j,i})$ in $L[X]$ for $j = 1, \ldots, m$. (Some numbers in the list $r_{j,1}, \ldots, r_{j,n_j}$ might be repeated.)

Using the roots $r_{j,i}$ of $f_1(X), \ldots, f_m(X)$, we can construct a ring homomorphism φ from $A = K[\{t_{f,i}\}]$ to L by substitution: φ fixes K, $\varphi(t_{f_j,i}) = r_{j,i}$ for $1 \leq i \leq n_j$, and $\varphi(t_{f,i}) = 0$ if f is not one of the f_j's. Extend φ to a homomorphism $A[X] \to L[X]$ by acting on coefficients and fixing X. The polynomial difference in (1) is mapped by φ to

$$f_j(X) - \prod_{i=1}^{n_j}(X - r_{f_j,i}) = 0 \text{ in } L[X],$$

and a polynomial in $L[X]$ is 0 only when all of its coefficients are 0. Therefore each coefficient of the powers of X in (1) is mapped by φ to 0 in L. In particular, $\varphi(c_j) = 0$. Thus φ sends the equation $1 = \sum_{j=1}^{m} a_j c_j$ in A to the equation $1 = 0$ in L, and that is a contradiction. $\square$

Since I is a proper ideal, Zorn's lemma guarantees that I is contained in some maximal ideal $\mathfrak{m}$ in A. (Probably I itself is not a maximal ideal, but I don't have a proof of that.) The quotient ring $A/\mathfrak{m} = K[\{t_{f,i}\}]/\mathfrak{m}$ is a field and the natural composite homomorphism $K \to A \to A/\mathfrak{m}$ of rings let us view the field $A/\mathfrak{m}$ as an extension of K (ring homomorphisms out of fields are always injective).

Theorem 2. *The field $A/\mathfrak{m}$ is an algebraic closure of K.*

Proof. For a nonconstant monic $f(X) \in K[X]$ we have $f(X) - \prod_{i=1}^{n_f}(X - t_{f,i}) \in I[X] \subset \mathfrak{m}[X]$, so in $(A/\mathfrak{m})[X]$ we have $f(X) = \prod_i (X - \bar{t}_{f,i})$, where $\bar{t}_{f,i}$ denotes $t_{f,i} \bmod \mathfrak{m}$. Each $\bar{t}_{f,i}$ is algebraic over K (being a root of $f(X)$) and A is generated as a ring over K by the $t_{f,i}$'s, so $A/\mathfrak{m}$ is generated as a ring over K by the $\bar{t}_{f,i}$'s. Therefore $A/\mathfrak{m}$ is an algebraic extension field of K in which every nonconstant monic in $K[X]$ splits completely.

We will now show $A/\mathfrak{m}$ is algebraically closed, and thus it is an algebraic closure of K. Set $F = A/\mathfrak{m}$. It suffices to show every monic irreducible $\pi(X)$ in $F[X]$ has a root in F. We have already seen that each nonconstant monic polynomial in $K[X]$ splits completely in $F[X]$, so let's show $\pi(X)$ is a factor of some monic polynomial in $K[X]$. There is a root α of $\pi(X)$ in some extension of F. Since α is algebraic over F and F is algebraic over K, α is algebraic over K. That implies some monic $f(X)$ in $K[X]$ has α as a root. The polynomial $\pi(X)$ is the minimal polynomial of α in $F[X]$, so $\pi(X) \mid f(X)$ in $F[X]$. Since $f(X)$ splits completely in $F[X]$, $\alpha \in F$. □

Our construction of an algebraic closure of K is done, but we want to compare it with another construction to put the one above in context.

The idea of building an algebraic closure of K by starting with a large polynomial ring over K whose variables are indexed by polynomials in $K[X]$ goes back at least to Emil Artin. He used a large polynomial ring (somewhat smaller than the ring $K[\{t_{f,i}\}]$ we started with above) modulo a suitable maximal ideal to obtain an algebraic extension K_1/K such that every nonconstant polynomial in $K[X]$ has a root in K_1 (not, *a priori*, that they all split completely in $K_1[X]$). Then he iterated this construction with K_1 in place of K to get a new algebraic extension K_2/K_1, and so on, and proved that the union $\bigcup_{n\geq 1} K_n$ (or, more rigorously, the direct limit of the K_n's) *contains* an algebraic closure of K [2, pp. 544-545]. With more work, treating separately characteristic 0 and characteristic p, it can be shown [3] that Artin's construction only needs one step: K_1 is an algebraic closure of K (so $K_n = K_1$ for all n, which is not obvious in Artin's own proof). In other words, the following is true: if F/K is an algebraic extension such that every nonconstant polynomial in $K[X]$ has a root in F then every nonconstant polynomial in $F[X]$ has a root in F, so F is an algebraic closure of K.

Theorem 2 and its proof, which I learned from B. Conrad, is a variation on a proof by Zorn [6] in the paper where he first introduced Zorn's lemma. It modifies Artin's construction by using a larger polynomial ring over K in order to adjoin to K in one step a full set of roots – not just one root – of each nonconstant monic in $K[X]$, rather than adjoining just one root for each nonconstant monic polynomial. Adjoining a full set of roots at once makes it easier to prove the constructed field $A/\mathfrak{m}$ is an algebraic closure of K. A similar construction, using a maximal ideal in a tensor product, is in [1, Prop. 4, p. A V 21].

Remark 3. That every field has an algebraic closure and that two algebraic closures of a field are isomorphic were first proved by Steinitz in 1910 in a long paper [5] that created from scratch the general theory of fields as part of abstract algebra. The influence of this paper on the development of algebra was enormous; for an indication of this, see [4]. Steinitz was hindered in his work by the primitive state of set theory at that time and he used the well-ordering principle rather than Zorn's lemma (which only became widely known in the 1930s [6]). Steinitz's proof of the existence of algebraic closures and their uniqueness up to isomorphism, together with his account of set theory, took up 20 pages [5, Sect. 19–21].

At the end of the proof of Theorem 2, the polynomial $f(X)$ in $K[X]$ with α as a root can be taken to be irreducible over K, so we could build an algebraic closure of K by defining the ideal I using just the monic *irreducible* $f(X)$ in $K[X]$ rather than all monic $f(X)$ in $K[X]$; the proofs of Lemma 1 and Theorem 2 carry over with no essential changes other than inserting the word "irreducible" in a few places. Finally, if we restrict the f in the construction of I to run over the monic separable polynomials in $K[X]$, or the monic separable irreducible polynomials in $K[X]$, then the field $A/\mathfrak{m}$ turns out to be a separable closure of K. The proof of Lemma 1 carries over with the f_j being separable (or separable irreducible), and in the proof of Theorem 2 two changes are needed: $A/\mathfrak{m}$ is a separable algebraic extension of K since it would be generated as a ring over K by roots of separable polynomials in $K[X]$, and we need transitivity of separability instead of algebraicity (if F/K is separable algebraic then each root of a separable polynomial in $F[X]$ is separable over K).

References

[1] N. Bourbaki, "Algebra II: Chapters 4-7," Springer-Verlag, New York, 1990.

[2] D. Dummit, R. Foote, "Abstract Algebra," 3rd ed., Wiley, New York, 2004.

[3] R. Gilmer, *A Note on the Algebraic Closure of a Field*, Amer. Math. Monthly **75** (1968), 1101-1102.

[4] P. Roquette, *In memoriam Ernst Steinitz* (1871–1928), J. Reine Angew. Math. **648** (2010), 1–11. URL `https://www.mathi.uni-heidelberg.de/~roquette/STEINITZ.pdf`.

[5] E. Steinitz, *Algebraische Theorie der Körper*, J. Reine Angew. Math. **137** (1910), 167–309. URL `https://eudml.org/doc/149323`.

[6] M. Zorn, *A Remark on a Method in Transfinite Algebra*, Bull. Amer. Math. Society **41** (1935), 667–670.