CONSTRUCTING ALGEBRAIC CLOSURES

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Let $K$ be a field. We want to construct an algebraic closure of $K$, i.e., an algebraic extension of $K$ which is algebraically closed. It will be built as the quotient of a polynomial ring in a very large number of variables.

For each nonconstant monic polynomial $f(X)$ in $K[X]$, let its degree be $n_f$ and let $t_{f,1}, \ldots, t_{f,n_f}$ be independent variables. Let $A = K[[t_{f,i}]]$ be the polynomial ring generated over $K$ by independent variables doubly indexed by every nonconstant monic $f \in K[X]$ and $1 \leq i \leq n_f$. This is a very large polynomial ring containing $K$.

Since $K \subset A$, we can view all $f(X)$ in $A[X]$. Let’s make them all split in some $(A/I)[X]$. Let $I$ be the ideal in $A$ generated by the coefficients of powers of $X$ in all the differences

$$f(X) - \prod_{i=1}^{n_f} (X - t_{f,i}) \in A[X]$$

as $f$ runs over nonconstant monic polynomials in $K[X]$. These differences all lie in $I[X]$, so $f(X) \equiv \prod_{i} (X - t_{f,i})$ in $(A/I)[X]$.

We want to use a maximal ideal in place of $I$ since working modulo a maximal ideal would give a complete splitting of all monic $f(X) \in K[X]$ over a field.

**Lemma 1.** The ideal $I$ is proper: $1 \notin I$.

**Proof.** We will argue by contradiction and the main idea is to use the existence of a splitting field for finitely many polynomials in $K[X]$.

Suppose $1 \in I$. Then we can write 1 as a finite sum $\sum_{j=1}^{m} a_j c_j$, where $a_j \in A$ and $c_j \in I$: each $c_j$ is a coefficient of some power of $X$ in some difference

$$f_j(X) - \prod_{i=1}^{n_j} (X - t_{j,i}),$$

where $n_j$ means $n_{f_j} = \deg f_j$. Since $f_j(X)$ is monic in $K[X]$ and $n_j = \deg(f_j)$, there is a (finite) field extension $L/K$ over which the polynomials $f_1(X), \ldots, f_m(X)$ all split completely, say $f_j(X) = \prod_{i=1}^{n_j} (X - r_{j,i})$ in $L[X]$ for $j = 1, \ldots, m$. (Some numbers in the list $r_{j,1}, \ldots, r_{j,n_j}$ might be repeated.)

Using the roots $r_{j,i}$ of $f_1(X), \ldots, f_m(X)$, we can construct a ring homomorphism $\varphi$ from $A = K[[t_{f,i}]]$ to $L$ by substitution: $\varphi$ fixes $K$, $\varphi(t_{f,i}) = r_{j,i}$ for $1 \leq i \leq n_j$, and $\varphi(t_{f_i}) = 0$ if $f$ is not one of the $f_j$’s. Extend $\varphi$ to a homomorphism $A[X] \rightarrow L[X]$ by acting on coefficients and fixing $X$. The polynomial difference in (1) is mapped by $\varphi$ to

$$f_j(X) - \prod_{i=1}^{n_j} (X - r_{j,i}) = 0 \text{ in } L[X],$$

and a polynomial in $L[X]$ is 0 only when all of its coefficients are 0. Therefore each coefficient of the powers of $X$ in (1) is mapped by $\varphi$ to 0 in $L$. In particular, $\varphi(c_j) = 0$. Thus $\varphi$ sends the equation $1 = \sum_{j=1}^{m} a_j c_j$ in $A$ to the equation $1 = 0$ in $L$, and that is a contradiction. \qed
Since $I$ is a proper ideal, Zorn’s lemma guarantees that $I$ is contained in some maximal ideal $m$ in $A$. (Probably $I$ itself is not a maximal ideal, but I don’t have a proof of that.) The quotient ring $A/m = K[[t_{f,i}]]/m$ is a field and the natural composite homomorphism $K \to A \to A/m$ of rings let us view the field $A/m$ as an extension of $K$ (ring homomorphisms out of fields are always injective).

**Theorem 2.** The field $A/m$ is an algebraic closure of $K$.

**Proof.** For a nonconstant monic $f(X) \in K[X]$ we have $f(X) - \prod_{i=1}^{n_f}(X - t_{f,i}) \in I[X] \subset m[X]$, so in $(A/m)[X]$ we have $f(X) = \prod_{i}(X - \bar{t}_{f,i})$, where $\bar{t}_{f,i}$ denotes $t_{f,i} \mod m$. Each $\bar{t}_{f,i}$ is algebraic over $K$ (being a root of $f(X)$) and $A$ is generated as a ring over $K$ by the $t_{f,i}$’s, so $A/m$ is generated as a ring over $K$ by the $\bar{t}_{f,i}$’s. Therefore $A/m$ is an algebraic extension field of $K$ in which every nonconstant monic in $K[X]$ splits completely.

We will now show $A/m$ is algebraically closed, and thus it is an algebraic closure of $K$. Set $F = A/m$. It suffices to show every monic irreducible $\pi(X)$ in $F[X]$ has a root in $F$. We have already seen that each nonconstant monic polynomial in $K[X]$ splits completely in $F[X]$, so let’s show $\pi(X)$ is a factor of some monic polynomial in $K[X]$. There is a root $\alpha$ of $\pi(X)$ in some extension of $F$. Since $\alpha$ is algebraic over $F$ and $F$ is algebraic over $K$, $\alpha$ is algebraic over $K$. That implies some monic $f(X)$ in $K[X]$ has $\alpha$ as a root. The polynomial $\pi(X)$ is the minimal polynomial of $\alpha$ in $F[X]$, so $\pi(X)\mid f(X)$ in $F[X]$. Since $f(X)$ splits completely in $F[X]$, $\alpha \in F$. \qed

Our construction of an algebraic closure of $K$ is done, but we want to compare it with another construction to put the one above in context.

The idea of building an algebraic closure of $K$ by starting with a large polynomial ring over $K$ whose variables are indexed by polynomials in $K[X]$ goes back at least to Emil Artin. He used a large polynomial ring (somewhat smaller than the ring $K[[t_{f,i}]]$ we started with above) modulo a suitable maximal ideal to obtain an algebraic extension $K_1/K$ such that every nonconstant polynomial in $K[X]$ has a root in $K_1$ (not, a priori, that they all split completely in $K_1[X]$). Then he iterated this construction with $K_1$ in place of $K$ to get a new algebraic extension $K_2/K_1$, and so on, and proved that the union $\bigcup_{n} K_n$ (or, more rigorously, the direct limit of the $K_n$’s) contains an algebraic closure of $K$ [2, pp. 544-545]. With more work, treating separately characteristic 0 and characteristic $p$, it can be shown [3] that Artin’s construction only needs one step: $K_1$ is an algebraic closure of $K$ (so $K_n = K_1$ for all $n$, which is not obvious in Artín’s own proof). In other words, the following is true: if $F/K$ is an algebraic extension such that every nonconstant polynomial in $K[X]$ has a root in $F$ then every nonconstant polynomial in $F[X]$ has a root in $F$, so $F$ is an algebraic closure of $K$.

Theorem 2 and its proof, which I learned from B. Conrad, is a variation on a proof by Zorn [6] in the paper where he first introduced Zorn’s lemma. It modifies Artin’s construction by using a larger polynomial ring over $K$ in order to adjoin to $K$ in one step a full set of roots – not just one root – of each nonconstant monic in $K[X]$, rather than adjoining just one root for each nonconstant monic polynomial. Adjoining a full set of roots at once makes it easier to prove the constructed field $A/m$ is an algebraic closure of $K$. A similar construction, using a maximal ideal in a tensor product, is in [1, Prop. 4, p. A V 21].

**Remark 3.** That every field has an algebraic closure and that two algebraic closures of a field are isomorphic were first proved by Steinitz in 1910 in a long paper that created from scratch the general theory of fields as part of abstract algebra. The influence of this paper
on the development of algebra was enormous; for an indication of this, see [4]. Steinitz was hindered in this work by the primitive state of set theory at that time and he used the well-ordering principle rather than Zorn’s lemma (which only became widely known in the 1930s [6]). Steinitz’s proof of the existence of algebraic closures and their uniqueness up to isomorphism, together with his account of set theory, took up 20 pages [5, Sect. 19–21].

At the end of the proof of Theorem 2, the polynomial \( f(X) \) in \( K[X] \) with \( \alpha \) as a root can be taken to be irreducible over \( K \), so we could build an algebraic closure of \( K \) by defining the ideal \( I \) using just the monic irreducible \( f(X) \) in \( K[X] \) rather than all monic \( f(X) \) in \( K[X] \); the proofs of Lemma 1 and Theorem 2 carry over with no essential changes other than inserting the word “irreducible” in a few places. Finally, if we restrict the \( f \) in the construction of \( I \) to run over the monic separable polynomials in \( K[X] \), or the monic separable irreducible polynomials in \( K[X] \), then the field \( A/\mathfrak{m} \) turns out to be a separable closure of \( K \). The proof of Lemma 1 carries over with the \( f_j \) being separable (or separable irreducible), and in the proof of Theorem 2 two changes are needed: \( A/\mathfrak{m} \) is a separable algebraic extension of \( K \) since it would be generated as a ring over \( K \) by roots of separable polynomials in \( K[X] \), and we need transitivity of separability instead of algebraicity (if \( F/K \) is separable algebraic then each root of a separable polynomial in \( F[X] \) is separable over \( K \)).

References