THE FUNDAMENTAL THEOREM OF ALGEBRA VIA PROPER MAPS

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1. Introduction

The Fundamental Theorem of Algebra says every nonconstant polynomial with complex coefficients can be factored into linear factors. The original form of this theorem makes no mention of complex polynomials or even complex numbers: it says that in $\mathbf{R}[x]$, every nonconstant polynomial can be factored into a product of linear and quadratic factors. (For any polynomial $f(x) \in \mathbf{C}[x]$, the product $f(x)\overline{f}(x)$ has real coefficients and this permits a passage between the real and complex formulations of the theorem.) That the theorem can be stated without complex numbers doesn't mean it can be proved without complex numbers, and indeed nearly all proofs of the Fundamental Theorem of Algebra make some use of complex numbers, either analytically (*e.g.*, holomorphic functions) or algebraically (*e.g.*, the only quadratic extension field of $\mathbf{R}$ is $\mathbf{C}$) or topologically (*e.g.*, $\mathrm{GL}_n(\mathbf{C})$ is path connected).

In the articles [3] and [4], Pukhlikov and Pushkar' give proofs of the Fundamental Theorem of Algebra that make absolutely no use of the concept of a complex number. Both articles are in Russian. The purpose of this note is to describe these two proofs in English so they may become more widely known.

As motivation for the two proofs, let's consider what it means to say a polynomial can be factored, in terms of the coefficients. To say that every polynomial $x^4+Ax^3+Bx^2+Cx+D$ in $\mathbf{R}[x]$ can be written as a product of two monic quadratic polynomials in $\mathbf{R}[x]$, so

$$\begin{aligned} x^4+Ax^3+Bx^2+Cx+D &= (x^2+ax+b)(x^2+cx+d) \\ &= x^4+(a+c)x^3+(b+d+ac)x^2+(ad+bc)x+bd, \end{aligned}$$

which amounts to saying that given any four real numbers A, B, C, D there is a real solution (a, b, c, d) to the system of equations

$$\begin{aligned} A &= a+c, \\ B &= b+d+ac, \\ C &= ad+bc, \\ D &= bd. \end{aligned}$$

This is a system of four (nonlinear) equations in four unknowns. Factoring a real polynomial of degree greater than 4 into lower-degree real factors involves more complicated constraints on the coefficients than the relations above, but they are also of the same basic flavor: a certain system of polynomial conditions in several real variables must have a real solution. Such systems of equations are not linear, so we can't prove they are solvable using linear algebra, and it may seem too complicated to prove directly that such "factorization equations" are always solvable. There was one very early proof of the Fundamental Theorem of Algebra, by Lagrange, which reasoned along these lines, although it is hard to argue that Lagrange's proof provides any conceptual insight (take a look at [5] and judge for yourself).

The basic idea in the two proofs presented here is to show the factorization constraints are solvable using topology. In the example above, for instance, we should show the map

$\mu\colon \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^4$ given by

$$\mu((a,b),(c,d)) = (a+c, b+d+ac, ad+bc, bd) \tag{1.1}$$

is surjective. There are theorems in topology that provide sufficient conditions for a continuous map to be surjective, and these theorems will lead to the two "real" proofs of the Fundamental Theorem of Algebra presented here.

In Section 2 we will review proper maps and describe the example of a proper map on polynomials that is common to the two proofs, which are developed separately in Sections 3 and 4.

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2. Proper Maps

For two topological spaces X and Y, a continuous map $f\colon X \to Y$ is called *proper* when the inverse image of any compact set is compact.

Example 2.1. Let $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$ be a nonconstant polynomial with real coefficients. It defines a continuous function $\mathbf{R} \to \mathbf{R}$. Let's show it is a proper map. In $\mathbf{R}$, a subset is compact when it is closed and bounded. If $K \subset \mathbf{R}$ is compact then $f^{-1}(K)$ is closed since K is closed and f is continuous. Since $|f(x)| \to \infty$ as $|x| \to \infty$, for any closed interval $[-R,R]$ the inverse image $f^{-1}([-R,R])$ is bounded. A compact set, such as K, is in some $[-R,R]$, so $f^{-1}(K)$ is bounded. Therefore $f^{-1}(K)$ is closed and bounded, so it is compact.

Example 2.2. Let $f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1z + a_0$ be a nonconstant polynomial with complex coefficients. It defines a continuous function $\mathbf{C} \to \mathbf{C}$. Since $|f(z)| \to \infty$ as $|z| \to \infty$, by the same argument as in the previous example f is a proper map.

Example 2.3. In contrast to the previous examples, polynomials in several variables need not be proper. The function $f\colon \mathbf{R}^2 \to \mathbf{R}$ given by $f(x,y) = xy$ is not proper since $\{0\}$ is compact but $f^{-1}(0)$ is the two coordinate axes, which is not compact. Similarly, the sine function $\sin\colon \mathbf{R} \to \mathbf{R}$ is not proper since $\sin^{-1}(0) = \pi\mathbf{Z}$ is not compact.

Example 2.4. If X is compact and Y is Hausdorff then any continuous map from X to Y is proper because a compact subset of a Hausdorff space is closed and a closed subset of a compact space is compact. In particular, all continuous mappings from one compact Hausdorff space to another are proper. This will be important for us later.

Remark 2.5. If X and Y are noncompact locally compact Hausdorff spaces and $X \cup \{x_\infty\}$ and $Y \cup \{y_\infty\}$ are the one-point compactifications of X and Y, a continuous function $f\colon X \to Y$ is proper when its extension $\widehat{f}\colon X \cup \{x_\infty\} \to Y \cup \{\infty\}$ given by $\widehat{f}(x_\infty) = y_\infty$ is continuous, so proper maps $X \to Y$ can be thought of as continuous functions that send "large" values to "large" values. We can see from this point of view why the functions $f(x,y) = xy$ and $f(x) = \sin x$ are not proper.

Here is the principal property we need about proper maps.

Theorem 2.6. *If X and Y are locally compact Hausdorff spaces, then any proper map $f\colon X \to Y$ has a closed image.*

Proof. First we give a "sequence" proof, which doesn't apply to general locally compact Hausdorff spaces but is valid for metrizable spaces, which will include the spaces we care about. Let $\{y_n\}$ be a sequence in $f(X)$ and assume $y_n \to y \in Y$. We want to show $y \in f(X)$. Let K be a compact neighborhood of y. Then $y_n \in K$ for $n \gg 0$. Write $y_n = f(x_n)$, so $x_n \in f^{-1}(K)$ for $n \gg 0$. Since $f^{-1}(K)$ is compact, there is a convergent subsequence $\{x_{n_i}\}$, say $x_{n_i} \to x \in f^{-1}(K)$. Since f is continuous, $f(x_{n_i}) \to f(x)$, so $y_{n_i} \to f(x)$. Since also $y_{n_i} \to y$, we get $y = f(x) \in f(X)$.

Next we give a general proof of the theorem, which will be quite different from the sequence proof. Let K be compact in Y. Then

$$f(X) \cap K = f(f^{-1}(K)),$$

which is compact in Y since f is proper. In a locally compact Hausdorff space, a subset that meets each compact set in a compact set is a closed subset. Therefore $f(X)$ is closed. □

Remark 2.7. In the notation of Theorem 2.6, if C is closed in X then $f|_C \colon C \to Y$ is proper and $(f|_C)(C) = f(C)$, so Theorem 2.6 implies $f(C)$ is closed. That is, a proper map between locally compact Hausdorff spaces is a closed map.

Lemma 2.8. *If $f \colon X \to Y$ is a proper map and B is any subset of Y then the restriction of f to a map $f^{-1}(B) \to B$ is proper.*

Proof. If $K \subset B$ is compact then $f^{-1}(K) \cap f^{-1}(B) = f^{-1}(K)$ is compact. □

Lemma 2.8 provides a method of showing a continuous function between non-compact spaces is proper: embed the non-compact spaces into compact spaces, check the original continuous function extends to a continuous function on the chosen compactification (where Example 2.4 might be used), and then return to the original function with Lemma 2.8.

Although polynomials in $\mathbf{R}[x]$ define proper maps from $\mathbf{R}$ to $\mathbf{R}$, this is *not* the way we will be using proper maps. We are going to use multiplication maps between spaces of polynomials with a fixed degree. For each positive integer d, let P_d be the space of monic polynomials of degree d:

$$x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0. \tag{2.1}$$

For $n \geq 2$ and $1 \leq k \leq n-1$, define the multiplication map

$$\mu_k \colon P_k \times P_{n-k} \to P_n \quad \text{by} \quad \mu_k(g,h) = gh.$$

For example, taking $n = 4$, $\mu_1 \colon P_1 \times P_3 \to P_4$ by

$$\mu_1(x + a_0, x^3 + b_2x^2 + b_1x + b_0) = x^4 + (b_2 + a_0)x^3 + (b_1 + a_0b_2)x^2 + (b_0 + a_0b_1)x + a_0b_0.$$

and $\mu_2 \colon P_2 \times P_2 \to P_4$ by

$$\mu_2(x^2 + a_1x + a_0, x^2 + b_1x + b_0) = x^4 + (a_1 + b_1)x^3 + (a_0 + b_0 + a_1b_1)x^2 + (a_1b_0 + b_1a_0)x + a_0b_0.$$

We met μ_2 in the introduction as the map $\mu \colon \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^4$ in (1.1).

To say a polynomial in P_n can be factored into polynomials of degree k and $n-k$ (without loss of generality the factors are monic too) amounts to saying the polynomial is in the image of μ_k.

To bring topology to bear on the study of μ_k, identify P_d with $\mathbf{R}^d$ by associating to the polynomial (2.1) the vector $(a_{d-1}, \ldots, a_1, a_0)$. This makes $\mu_k \colon P_k \times P_{n-k} \to P_n$ a continuous mapping between two locally compact spaces.

Theorem 2.9. *For $1 \leq k \leq n-1$, the mapping $\mu_k \colon P_k \times P_{n-k} \to P_n$ is proper.*

Proof. Using coefficients to identify P_d with $\mathbf{R}^d$, μ_k is a continuous mapping $\mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^n$. Since these spaces are not compact, properness of μ_k is not obvious. To make it obvious, we will follow an idea of A. Khovanskii and "compactify" μ_k to a mapping between compact spaces, to which we we can apply Example 2.4.

Rather than looking at the spaces P_d, let's consider *all* the nonzero polynomials of degree $\leq d$ and identify two such polynomials if they are scalar multiples of one another. Call this set $\widehat{P}_d$ and write the equivalence class of a polynomial f in $\widehat{P}_d$ as $[f]$. For example,

$[x^2+1] = [3x^2+3]$ and $[2x^3 - x + 4] = [x^3 - (1/2)x + 2]$. Every equivalence class in $\widehat{P}_d$ contains a unique monic polynomial of degree at most d, since a monic polynomial can't be scaled by a number other than 1 and remain monic. Therefore P_d embeds into $\widehat{P}_d$ by $f \mapsto [f]$. The image of P_d in $\widehat{P}_d$ is the equivalence classes of polynomials with exact degree d (not less than d).

We can identify $\widehat{P}_d$ with real projective d-space $\mathbf{P}^d(\mathbf{R})$ by associating to the equivalence class of polynomials $[a_dx^d + \cdots + a_1x + a_0]$ the point $[a_d, \ldots, a_1, a_0]$. Since $\mathbf{P}^d(\mathbf{R})$ is a compact Hausdorff space, $\widehat{P}_d$ becomes a compact Hausdorff space and the copy of P_d inside $\widehat{P}_d$ is identified with a standard copy of $\mathbf{R}^d$ in $\mathbf{P}^d(\mathbf{R})$: the points whose first homogeneous coordinate is not 0. From a projective point of view, the polynomials "at infinity" in $\widehat{P}_d$ are those with degree less than d.

Since polynomials that are determined up to an overall (nonzero) scaling factor have a product that is determined up to an overall scaling factor, we can define a multiplication map $\widehat{\mu}_k \colon \widehat{P}_k \times \widehat{P}_{n-k} \to \widehat{P}_n$ by $\widehat{\mu}_k([g],[h]) = [gh]$. For example, when $n = 3$ and $k = 1$,

$$\mu_1([a_1x + a_0], [b_2x^2 + b_1x + b_0]) = [a_1b_2x^3 + (a_1b_1 + a_0b_2)x^2 + (a_1b_0 + a_0b_1)x + a_0b_0].$$

Viewing each P_d inside of $\widehat{P}_d$, the restriction of $\widehat{\mu}_k$ from $\widehat{P}_k \times \widehat{P}_{n-k}$ to $P_k \times P_{n-k}$ is the mapping $\mu_k \colon P_k \times P_{n-k} \to P_n$ defined earlier. In projective coordinates, $\widehat{\mu}_k$ is a polynomial mapping so it is continuous. Since projective spaces are compact and Hausdorff, $\widehat{\mu}_k$ is a proper map by Example 2.4. Finally, since $\widehat{\mu}_k^{-1}(P_n) = P_k \times P_{n-k}$, Lemma 2.8 tells us μ_k is proper. □

Corollary 2.10. *If $n = d_1 + \cdots + d_r$, where each d_i is a positive integer, the map $\mu \colon P_{d_1} \times \cdots \times P_{d_r} \to P_n$ given by $\mu(f_1, \ldots, f_r) = f_1 \cdots f_r$ is proper.*

Proof. This proceeds in the same way as the proof of Theorem 2.9, using a multiplication map $\widehat{\mu} \colon \widehat{P}_{d_1} \times \cdots \times \widehat{P}_{d_r} \to \widehat{P}_n$ that extends the map μ. □

For each k from 1 to $n-1$, the set of polynomials in P_n that can be written as a product of (monic) polynomials of degree k and $n-k$ is $\mu_k(P_k \times P_{n-k})$, which is closed by Remark 2.7 and Theorem 2.9. For example, a fifth degree monic polynomial in $\mathbf{R}[x]$ that is the coefficientwise limit of a sequence of fifth degree monic polynomials that each factor as a quadratic and cubic will itself factor as a quadratic and cubic. More generally, Remark 2.7 and Corollary 2.10 imply that the polynomials in P_n that admit some factorization with fixed degrees $d_1, \ldots, d_r$ for the factors form a closed subset of P_n. That means polynomial factorizations respect limit operations: if $f_i \to f$ in P_n and each f_i is a product of monic polynomials of degree $d_1, d_2, \ldots, d_r$ then f also has such a factorization. This is nontrivial since convergence of polynomials conveys no direct information about how factorizations behave along the way.[1]

3. Proof of Pukhlikov

For the first proof we give of the Fundamental Theorem of Algebra, due to Pukhlikov [3], we want to show for $n \geq 1$ that each polynomial in P_n can be written as a product of linear and quadratic polynomials in $\mathbf{R}[x]$. The argument is by induction on n, and since it is clear when n is 1 or 2 we take $n \geq 3$ from now on. Assume by induction that every monic polynomial in the spaces $P_1, \ldots, P_{n-1}$ is a product of linear and quadratic real polynomials. We will show every monic polynomial in P_n is a product of a polynomial in some P_k and

[1]The proofs of Theorem 2.9 and Corollary 2.10 remain true when $\mathbf{R}$ is replaced by any locally compact field F, like $\mathbf{C}$ or a p-adic field, so the monic polynomials of degree n in $F[x]$ with a factorization in $F[x]$ having fixed degrees form a closed subset of all monics of degree n in $F[x]$.

P_{n-k} where $1 \leq k \leq n-1$ and therefore is a product of linear and quadratic polynomials. Using the multiplication maps μ_k, what we want to show is

$$P_n = \bigcup_{k=1}^{n-1} \mu_k(P_k \times P_{n-k}).$$

Set $Z_k = \mu_k(P_k \times P_{n-k})$ and

$$Z = \bigcup_{k=1}^{n-1} Z_k.$$

The set Z is all the monic polynomials of degree n that are reducible (and thus are products of linear and quadratic polynomials by the inductive hypothesis). We want to show $Z = P_n$.

Since each μ_k is proper, its image Z_k is a closed subset of P_n. Since $Z = Z_1 \cup \cdots \cup Z_{n-1}$ is a finite union of closed sets, Z is closed in P_n. Topologically, $P_n \cong \mathbf{R}^n$ is *connected*, so if we could show Z is also open in P_n then we would immediately get $Z = P_n$ (since $Z \neq \emptyset$), which is the goal. Alas, it will not be easy to show Z is open directly, but a modification of this idea will work.

To say Z is open in P_n means for any polynomial f in Z, all polynomials in P_n that are near f are also in Z. The inverse function theorem is a natural tool to use here: supposing $f = \mu_k(g, h)$, is the Jacobian determinant of $\mu_k \colon P_k \times P_{n-k} \to P_n$ nonzero at (g, h)? If it is, then μ_k has a continuous local inverse defined in a neighborhood of f.

To analyze μ_k near (g, h), we can write all nearby points in $P_k \times P_{n-k}$ as $(g + u, h + v)$ where $\deg u \leq k - 1$ and $\deg v \leq n - k - 1$ (allowing $u = 0$ or $v = 0$ too). Then

$$\mu_k(g + u, h + v) = (g + u)(h + v) = gh + gv + hu + uv = f + (gv + hu) + uv. \tag{3.1}$$

As functions of the coefficients of u and v, the coefficients of $gv + hu$ are all *linear* and the coefficients of uv are all higher degree polynomials in the coefficients of u and v. Whether the Jacobian of μ_k at (g, h) is invertible or not depends on the uniqueness of writing polynomials with degree less than n as $gv + hu$ for some u and v.

Lemma 3.1. *Let g and h be nonconstant polynomials whose degrees add up to n.*

a) *If g and h are relatively prime then every polynomial of degree less than n is uniquely of the form $gv + hu$ where $\deg u < \deg g$ or $u = 0$, and $\deg v < \deg h$ or $v = 0$.*
b) *If g and h are not relatively prime then we can write $gv + hu = 0$ for some nonzero polynomials u and v where $\deg u < \deg g$ and $\deg v < \deg h$.*

Proof. a) By counting dimensions, it suffices to show the only way to write $gv + hu = 0$ with $\deg u < \deg g$ or $u = 0$ and $\deg v < \deg h$ or $v = 0$ is by using $u = 0$ and $v = 0$. Since $gv = -hu$, $g|hu$, so $g|u$ since g and h are relatively prime. If $u \neq 0$ then we get a contradiction from the inequality $\deg u < \deg g$. Therefore $u = 0$, so also $v = 0$.

b) Let $d(x)$ be a nonconstant common factor of $g(x)$ and $h(x)$. Write $g(x) = d(x)a(x)$ and $h(x) = d(x)b(x)$. Then $g(x)b(x) + h(x)(-a(x)) = 0$. Set $u(x) = b(x)$ and $v(x) = -a(x)$. □

By (3.1) and Lemma 3.1, the Jacobian matrix of μ_k at (g, h) is invertible if g and h are relatively prime and not otherwise.[2] We conclude that if $f \in Z$ can somehow be written as a product of nonconstant relatively prime polynomials of degrees k and $n - k$ then all monic polynomials in a neighborhood of f in P_n can be factored in the same way, so this neighborhood is inside Z. Which $f \in Z$ *can't* be written as a product of nonconstant relatively prime

[2]The Jacobian determinant of μ_k at (g, h) is equal to the *resultant* of g and h, so we recover the classical theorem that the resultant of two polynomials is nonzero exactly when they are relatively prime, which is the same as saying they have no common root in a splitting field.

polynomials?[3] Every monic polynomial in $\mathbf{R}[x]$ has a monic factorization into irreducibles in $\mathbf{R}[x]$, and as soon as a polynomial has two different monic irreducible factors it has a decomposition into nonconstant relatively prime factors. Therefore a polynomial $f \in Z$ is not a product of at least 2 nonconstant relatively prime polynomials in $\mathbf{R}[x]$ exactly when it is a power of a monic irreducible in $\mathbf{R}[x]$. Since each polynomial in Z is a product of linear and quadratic factors, f is a power of a linear or quadratic polynomial. Let Y be all these "degenerate" polynomials in P_n:

$$Y = \begin{cases} \{(x+a)^n : a \in \mathbf{R}\} \text{ if } n \text{ is odd}, \\ \{(x+a)^n, (x^2+bx+c)^{n/2} : a, b, c \in \mathbf{R}\} \text{ if } n \text{ is even}. \end{cases}$$

When n is even, we can write $(x+a)^n$ as $(x^2+2ax+a^2)^{n/2}$, so[4]

$$Y = \begin{cases} \{(x+a)^n : a \in \mathbf{R}\} \text{ if } n \text{ is odd}, \\ \{(x^2+bx+c)^{n/2} : b, c \in \mathbf{R}\} \text{ if } n \text{ is even}. \end{cases}$$

We have shown $Z - Y$ is open in P_n. This is weaker than the plan to show Z is open in P_n. But we're actually in good shape, as long as we change the focus from P_n to $P_n - Y$. If $n = 2$ then $Y = P_2$ and $P_2 - Y$ is empty. For the first time we will use the fact that $n \geq 3$.

Lemma 3.2. *If $n \geq 3$, then $P_n - Y$ is path connected.*

Proof. We identify P_n with $\mathbf{R}^n$ using polynomial coefficients:

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mapsto (a_{n-1}, \ldots, a_1, a_0).$$

If n is odd then Y is a smooth curve sitting in $\mathbf{R}^n$: it is the image of the polynomial map $\mathbf{R} \to \mathbf{R}^n$ that associates to $a \in \mathbf{R}$ the non-leading coefficients of $(x+a)^n$ in the order of decreasing degree. For instance, if $n = 3$ this map is

$$a \mapsto (3a, 3a^2, a^3).$$

If n is even then Y is a smooth surface in $\mathbf{R}^n$. For instance, if $n = 4$ then computing $(x^2+bx+c)^2$ shows us that

$$Y = \{(2b, b^2+2c, 2bc, c^2) : b, c \in \mathbf{R}\} \subset \mathbf{R}^4,$$

which is smooth since the 2×4 matrix of partial derivatives at a point is

$$\begin{pmatrix} 2 & 2b & 2c & 0 \\ 0 & 2 & 2b & 2c \end{pmatrix},$$

which has rank 2 from its first two columns.

It is left to the reader to check that the complement of a smooth[5] curve in $\mathbf{R}^n$ is path connected for $n \geq 3$ and the complement of a smooth surface in $\mathbf{R}^n$ is path connected for $n \geq 4$. $\square$

When $n \geq 3$, $(x-1)(x-2)\cdots(x-n) \in Z - Y$, so $Z - Y$ is nonempty. Since Z is closed in P_n, $Z \cap (P_n - Y) = Z - Y$ is closed in $P_n - Y$. Since we know $Z - Y$ is open in P_n, it is open in $P_n - Y$. Therefore $Z - Y$ is a nonempty open and closed subset of $P_n - Y$. Since $P_n - Y$

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[3] Viewing $\mu_k \colon P_k \times P_{n-k} \to P_k$ as a smooth map of manifolds, such f are not just critical values of μ_k: *all* points in $\mu_k^{-1}(f)$ have a noninvertible differential. Such f are called strongly critical values in [4].

[4] Since any odd-degree real polynomial has a real root, obviously $P_n = \mu_1(P_1 \times P_{n-1})$, so we could just focus on even n.

[5] Space-filling curves show some constraint is necessary for the complement of a "curve" in $\mathbf{R}^n$ to be path connected.

is path connected, and thus connected,[6] and $Z - Y$ is not empty, $Z - Y = P_n - Y$. Since $Y \subset Z$, we get $Z = P_n$ and this completes the first proof of the Fundamental Theorem of Algebra.

4. Proof of Pushkar'

The second proof of the Fundamental Theorem of Algebra, by Pushkar' [4], focuses on even degree polynomials. (Any real polynomial of odd degree has a real root.) For any $n \geq 1$, we want to show each polynomial in P_{2n} can be factored into a product of n monic quadratic polynomials. Consider the multiplication mapping

$$u_n \colon P_2^n \to P_{2n} \quad \text{where} \quad u_n(f_1, f_2, \ldots, f_n) = f_1 f_2 \cdots f_n. \tag{4.1}$$

To prove the Fundamental Theorem of algebra, we'll show u_n is surjective for all even $n \geq 2$. By Corollary 2.10, u_n is proper.

We will use a theorem from topology about the degree of a mapping. For a smooth proper mapping $\varphi \colon M \to N$ of connected oriented manifolds M and N with the same dimension, the *degree* of φ is defined by picking any regular value[7] y and forming

$$\sum_{x \in \varphi^{-1}(y)} \varepsilon_x,$$

where $\varepsilon_x = 1$ is $d\varphi_x \colon T_x(M) \to T_y(N)$ is orientation preserving and $\varepsilon_x = -1$ if $d\varphi_x$ is orientation-reversing. If $\varphi^{-1}(y)$ is empty, set this sum to be 0. That the degree is well-defined is rather complicated to prove. Details can be found in [1, Sect. 13]. It immediately implies the following result.

Theorem 4.1. *Let M and N be smooth connected oriented manifolds of the same dimension and $\varphi \colon M \to N$ be a smooth proper mapping of degree not equal to zero. Then f is surjective.*

We will apply Theorem 4.1 to the mapping $u_n \colon P_2^n \to P_{2n}$ in (4.1). Choose an orientation for P_2, which is naturally identified with $\mathbf{R}^2$, and then give P_2^n the product orientation (as a product of oriented manifolds). Orient P_{2n} by identifying it with $\mathbf{R}^{2n}$ in a natural way. We want to show u_n has nonzero degree. To do this, we need a regular value.

Lemma 4.2. *The polynomial $p(x) = \prod_{i=1}^{n}(x^2 + i) = u_n(x^2 + 1, \ldots, x^2 + n)$ in P_{2n} is a regular value of u_n.*

Proof. Exercise. Note $p(x)$ is a product of distinct monic quadratic irreducibles. Look at the description of the regular points of the multiplication mappings μ_k in Section 3. □

The polynomial $p(x)$ has $n!$ inverse images under u_n: all ordered n-tuples with coordinates $x^2 + i$ for $i = 1, \ldots, n$. Since u_n is invariant under permutations of its arguments, and any such permutation preserves orientation (exercise), the sign of the determinant of du_n at each point in $u_n^{-1}(p(x))$ is the same. Since these points each contribute the same sign to the degree, u_n has degree $n!$ or $-n!$ (the exact choice depends on how we oriented P_2 and P_{2n}). Since the degree is not zero, by Theorem 4.1 u_n is surjective, which completes the second proof of the Fundamental Theorem of Algebra.

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[6] If we had removed from P_n not Y but the larger set of nonseparable polynomials, we would not be left with a connected set: the nonseparable polynomials in P_n divide the rest of P_n into two open subsets: those with positive discriminant and those with negative discriminant. However, if we were working over $\mathbf{C}$ instead of over $\mathbf{R}$ then the complement of the nonseparable polynomials would be connected. An analogue of Pukhlikov's "real" proof that is carried out over $\mathbf{C}$ was given by Litt [2].

[7] A regular value is a point $y \in N$ such that for all $x \in \varphi^{-1}(y)$, $d\varphi_x \colon T_x(M) \to T_y(N)$ is surjective. This includes any y for which $\varphi^{-1}(y)$ is empty. Regular values in N exist by Sard's theorem.

The novel feature of these two proofs of the Fundamental Theorem of Algebra is the topological study of multiplication maps on real polynomials to settle a factorization question. Complex numbers never enter the argument, even indirectly. The proofs provide an interesting topic for discussion in a course on differential topology, on account of the ideas that are used: spaces of real polynomials with fixed or bounded degree as an abstract manifold, passage to a projective space to "compactify" a smooth map and show it is proper, determining where the inverse function theorem can be applied, and the degree of a map. That proper maps and the degree of a map can be used to prove the Fundamental Theorem of Algebra is not itself new: proofs using these ideas appear in complex variable proofs. Here is one such proof, taken from [1, p. 109].

Proof. Pick any nonconstant monic polynomial with complex coefficients, say

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

We want to show f has a complex zero. As a continuous mapping $\mathbf{C} \to \mathbf{C}$, f is proper (Example 2.2). Its degree is n since f is smoothly homotopic to $z \mapsto z^n$, whose degree is easy to calculate as n. We now find ourselves in the situation of Theorem 4.1, which proves f is surjective, so f has a complex root. □

References

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