THE FUNDAMENTAL THEOREM OF ALGEBRA VIA MULTIVARIABLE CALCULUS

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This is a proof of the fundamental theorem of algebra which is due to Gauss [2], in 1816. It is based on [1, pp. 680–682]. The proof is accessible, in principle, to anyone who has had multivariable calculus and knows about complex numbers. The main idea will be to compute a certain double integral and then compute the integral in the other order.

We take for granted the following result from calculus, which is a special case of Fubini’s theorem.

Lemma 1. Let \([a,b] \times [c,d] \subset \mathbb{R}^2\) be a rectangle, and \(f\) be a continuous function on this rectangle, with real values. Then

\[
\int_c^d \left( \int_a^b f(x,y) \, dx \right) dy = \int_a^b \left( \int_c^d f(x,y) \, dy \right) dx.
\]

Theorem 1. Every nonconstant polynomial in \(\mathbb{C}[z]\) has a complex root.

Proof. We are going to prove the contrapositive: if

\[f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0\]

has no complex roots, then \(f(z)\) is a (nonzero) constant. Here \(n\) is the degree of \(f\).

Write \(z = re^{i\theta}\), so \(z^j = r^j \cos(j\theta) + ir^j \sin(j\theta)\). Therefore the decomposition of \(f(z)\) into real and imaginary parts is

\[f(z) = P(r,\theta) + iQ(r,\theta),\]

where

\[P(r,\theta) = r^n \cos(n\theta) + \cdots + \text{Re}(c_0), \quad Q(r,\theta) = r^n \sin(n\theta) + \cdots + \text{Im}(c_0).\]

Both \(P\) and \(Q\) are polynomials in \(r\) of degree \(n\), with constant terms independent of \(\theta\). (In particular, a trigonometric function of \(\theta\) appears in \(P\) and \(Q\) only when multiplied by positive powers of \(r\), so the ambiguity in the definition of \(\theta\) at the origin does not matter: \(P(0,\theta) = \text{Re}(c_0)\) and \(Q(0,\theta) = \text{Im}(c_0)\) for all \(\theta\).) From this observation about the constant terms,

\[
\frac{\partial P}{\partial \theta} \bigg|_{r=0} = 0, \quad \frac{\partial Q}{\partial \theta} \bigg|_{r=0} = 0.
\]

Clearly \(P\) and \(Q\) are 2\(\pi\)-periodic, as are \(\partial P/\partial r\) and \(\partial Q/\partial r\).

To say \(f\) has no complex roots is the same as saying \(P\) and \(Q\) are not simultaneously 0 anywhere. Writing \(f(z) = P + iQ\) in polar coordinates, we contemplate its angular component, \(\text{arctan}(Q/P)\).

Set

\[U = \arctan \left( \frac{Q}{P} \right).\]

From the derivative formula for the arctangent,

\[
\frac{\partial U}{\partial r} = \frac{1}{(1 + Q/P)^2} \cdot \frac{PQ_r - QP_r}{P^2} = \frac{PQ_r - QP_r}{P^2 + Q^2}
\]
\[
\frac{\partial U}{\partial \theta} = \frac{PQ_\theta - QP_\theta}{P^2 + Q^2},
\]
where we adopt the subscript notation for partial derivatives. The formulas on the right side of (1) and (2) make sense everywhere, since \(P^2 + Q^2 \neq 0\) for all \((r, \theta)\). However, there is something mysterious about the definition of the function \(U\) as a “value” of \(\arctan\). Usually one defines the function \(\arctan x\) to take values in \((-\pi/2, \pi/2)\), with values \(\pm \pi/2\) at \(\pm \infty\) from the asymptotics visible on the graph of \(y = \arctan x\). But this kind of definition is bad to use in the definition of \(U\), because we can imagine wandering through a point in the plane where \(P = 0\) (and thus where \(Q/P\) is “infinite”) such that the continuous variation in \(\arctan\) may demand that the function \(U\) increase above the value \(\pi/2\).

This is the same kind of problem one meets when trying to define logarithms of complex numbers, but we can circumvent the trouble with \(U\) by taking the right sides of (1) and (2) as our basic functions (i.e., the partial derivative notation on the left sides is purely suggestive, at least for readers who only know up to multivariable calculus). For example, the formula
\[
\frac{\partial}{\partial \theta} \left( \frac{\partial U}{\partial r} \right) = \frac{\partial}{\partial r} \left( \frac{\partial U}{\partial \theta} \right)
\]
can be checked by a direct calculation of the \(\theta\)-partial of the right side of (1) and the \(r\)-partial of the right side of (2). We do not appeal to the theorem on equality of mixed partials. The common “iterated” derivative in (3) has the form \(H(r, \theta)/(P^2 + Q^2)^2\) for an explicit continuous function \(H\).

Applying Lemma 1 to the rectangle \([0, R] \times [0, 2\pi]\) (with \(R > 0\)) in the \((r, \theta)\) plane, and integrating the function in (3), we have
\[
\int_0^R \left( \int_0^{2\pi} \frac{\partial}{\partial r} \frac{\partial U}{\partial \theta} \, d\theta \right) \, dr = \int_0^{2\pi} \left( \int_0^R \frac{\partial}{\partial r} \frac{\partial U}{\partial \theta} \, dr \right) \, d\theta.
\]
On the left side, we evaluate the inner integral by appealing to (3):
\[
\int_0^{2\pi} \frac{\partial}{\partial r} \frac{\partial U}{\partial \theta} \, d\theta = \int_0^{2\pi} \frac{\partial}{\partial \theta} \frac{\partial U}{\partial r} \, d\theta = \frac{\partial U}{\partial \theta} \bigg|_{\theta=2\pi} - \frac{\partial U}{\partial \theta} \bigg|_{\theta=0} = 0,
\]
since \(\partial U/\partial r\) is \(2\pi\)-periodic. Therefore the left side of (4) is 0 for all \(R > 0\).

Now we compute the right side of (4). The inside integral is
\[
\int_0^R \frac{\partial}{\partial r} \frac{\partial U}{\partial \theta} \, dr = \left. \frac{\partial U}{\partial \theta} \right|_{r=0} - \left. \frac{\partial U}{\partial \theta} \right|_{r=R} = 0,
\]
since the \(\theta\)-partials of \(P\) and \(Q\) vanish at \(r = 0\). Having separately computed the two sides of (4), we conclude that for \(R > 0\),
\[
\left. \frac{\partial U}{\partial \theta} \right|_{r=R} = 0.
\]
Now we are going to compute the value of this partial derivative by the explicit formula (2). First we look at the numerator. Because
\[
P_\theta = -nr^n \sin(n\theta) + \cdots, \quad Q_\theta = nr^n \cos(n\theta) + \cdots,
\]
where \( \cdots \) represents terms of lower degree in \( r \),
\[
PQ_\theta - QP_\theta = nr^{2n}\cos^2(n\theta) + \cdots + nr^{2n}\sin(n\theta) + \cdots = nr^{2n} + \cdots.
\]
Similarly, the denominator in (2) is \( r^{2n} + \cdots \), so
\[
\frac{\partial U}{\partial \theta} = nr^{2n} + \cdots r^{2n} + \cdots.
\]
The lower degree terms have \( \theta \) appearing only inside trigonometric (and thus bounded) functions, hence
\[
\lim_{R \to \infty} \left. \frac{\partial U}{\partial \theta} \right|_{r=R} = n
\]
uniformly in \( \theta \). That lets us evaluate the right side of (4), and obtain
\[
\int_0^{2\pi} \left( \int_0^R \frac{\partial}{\partial r} \frac{\partial U}{\partial \theta} \, dr \right) \, d\theta = \int_0^{2\pi} \left. \frac{\partial U}{\partial \theta} \right|_{r=R} \, d\theta \to 2\pi n
\]
as \( R \to \infty \). On the other hand, we already computed from (4) that the integral is 0. Therefore \( n = 0 \).

To summarize the argument, we showed that if \( f(z) \) has degree \( n \) and \( f(z) \neq 0 \) for all complex \( z \), then \( U = \arctan(\text{Im}(f)/\text{Re}(f)) \) satisfies
\[
0 = \int_0^R \left( \int_0^{2\pi} \frac{\partial}{\partial r} \frac{\partial U}{\partial \theta} \, d\theta \right) \, dr = \int_0^{2\pi} \left( \int_0^R \frac{\partial}{\partial r} \frac{\partial U}{\partial \theta} \, dr \right) \, d\theta \to 2\pi n
\]
as \( R \to \infty \), so \( n = 0 \), i.e., \( f \) is a constant.

Since \( \arctan(Q/P) \) is essentially the argument of \( P + iQ = f \), this proof of Gauss is a precursor of the proof of the Fundamental Theorem of Algebra based on winding numbers, which involves the computation of \( (1/2\pi i) \int_C (f'(z)/f(z)) \, dz \).

REFERENCES