ESTIMATING THE SIZE OF A DIVERGENT SUM

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Estimation is a fundamental skill in analysis. It is more often used as a tool to prove a sequence or series converges, but it is equally useful for divergent sequences or series. Indeed, sometimes it is important to accurately estimate the rate of divergence. As a simple example, the divergence of the harmonic series $1 + 1/2 + 1/3 + \cdots$ is understood better when we are aware that it diverges logarithmically:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \log n.$$

In this handout, we discuss a practical method of getting good estimates on divergent series, based on a good understanding of rates of growth.

Two sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ of positive real numbers will be called *asymptotic* if

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 1.$$

We then write $x_n \sim y_n$. For example, $\sqrt{n^2+2} \sim n$ and $\frac{1}{n+3} \sim \frac{1}{n}$. If $x_n \sim y_n$, then $1/2 < x_n/y_n < 3/2$ for large n, so $x_n \to \infty$ if and only if $y_n \to \infty$.

We show that the partial sums of asymptotic sequences are again asymptotic, provided (either of) the sums diverge.

Theorem 1. *If $x_n \sim y_n$ and either sequence of partial sums, $\{\sum_{k=1}^n x_k\}$ or $\{\sum_{k=1}^n y_k\}$, tends to ∞ as $n \to \infty$, then both partial sum sequences tend to ∞ and the partial sum sequences are asymptotic.*

Proof: Fix $\varepsilon > 0$. For all k large enough, say for $k > N$, $|x_k/y_k - 1| \leq \varepsilon$, so

$$(1-\varepsilon)y_k \leq x_k \leq (1+\varepsilon)y_k.$$

Thus, for $n > N$,

$$(1-\varepsilon)\sum_{k=N+1}^{n} y_k \leq \sum_{k=N+1}^{n} x_k \leq (1+\varepsilon)\sum_{k=N+1}^{n} y_k,$$

so

$$1 - \varepsilon \leq \frac{\sum_{k=N+1}^n x_k}{\sum_{k=N+1}^n y_k} \leq 1 + \varepsilon. \tag{1}$$

Since ε and N are fixed, we see by letting $n \to \infty$ in (1) that $\sum_{k=1}^n x_k \to \infty$ if and only if $\sum_{k=1}^n y_k \to \infty$, so by our hypothesis both partial sum sequences tend to infinity.

To simplify the notation, let $X_n = \sum_{k=1}^n x_k$ and $Y_n = \sum_{k=1}^n y_k$. We want to show $X_n \sim Y_n$. For all $n > N$,

$$\frac{X_n}{Y_n} = \frac{X_N + (X_n - X_N)}{Y_N + (Y_n - Y_N)}.$$

By (1), $1-\varepsilon \le \frac{X_n - X_N}{Y_n - Y_N} \le 1+\varepsilon$. So dividing the numerator and denominator of the right hand side of the above equation by $Y_n - Y_N > 0$, we see that

$$\frac{\frac{X_N}{Y_n - Y_N} + 1 - \varepsilon}{\frac{Y_N}{Y_n - Y_N} + 1} \le \frac{X_n}{Y_n} \le \frac{\frac{X_N}{Y_n - Y_N} + 1 + \varepsilon}{\frac{Y_N}{Y_n - Y_N} + 1}$$

for $n > N$. As $n \to \infty$, $Y_n - Y_N \to \infty$, so

$$1 - \varepsilon \le \liminf \frac{X_n}{Y_n} \le \limsup \frac{X_n}{Y_n} \le 1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we're done.

□

Example 1. The growth of the sums $\sum_{k=1}^n \sqrt{k^2+k}$ is just like that of $\sum_{k=1}^n k$, since $\sqrt{k^2+k} \sim k$. We know, explicitly, that $\sum_{k=1}^n k = (n^2+n)/2$, which is asymptotic to $n^2/2$, so

$$\sum_{k=1}^n \sqrt{k^2+k} \sim \frac{n^2}{2}.$$

It is not clear how to apply Theorem 1 to get a good estimate on, say, $\sum_{k=1}^n \sqrt{k}$ or $\sum_{k=1}^n 1/\sqrt{k}$. To make the asymptotic estimation of divergent sums feasible, we link them with integrals. The idea here is that most natural sequences one meets are the values of a continuous function at integers (*e.g.*, $\sqrt{x}$ or $1/\sqrt{x}$). The sum of this function over the integers from 1 to n behaves like the integral of the function over $[1, n]$, when $n \to \infty$, as long as the function is eventually monotonic, *i.e.*, is either increasing for large x or decreasing for large x.

Theorem 2. *Let $f(x)$ be a continuous function which is either increasing for all large x or decreasing for all large x. Provided $f(x) \sim f(x+1)$ as $x \to \infty$ and $\int_1^n f(x)\,\mathrm{d}x \to \infty$ as $n \to \infty$,*

$$\sum_{k=1}^n f(k) \sim \int_1^n f(x)\,\mathrm{d}x.$$

Before we prove the theorem, let's see how it can be used in two examples.

Example 2. We estimate $\sum_{k=1}^n \sqrt{k}$. The function involved here is, naturally, $f(x) = \sqrt{x}$. Does it fit the hypotheses of Theorem 2? This function is increasing for $x > 0$. It is clear that $\sqrt{x+1} \sim \sqrt{x}$ as $x \to \infty$. Finally, $\int_1^n \sqrt{x}\,\mathrm{d}x \to \infty$ since $\sqrt{x} \to \infty$. Therefore, Theorem 2 tells us that

$$\sum_{k=1}^n \sqrt{k} \sim \int_1^n \sqrt{x}\,\mathrm{d}x = \frac{2}{3}\left(n^{3/2} - 1\right) \sim \frac{2}{3} n^{3/2}.$$

Example 3. We estimate $\sum_{k=1}^n 1/\sqrt{k}$. Now $f(x) = 1/\sqrt{x}$. Does it fit the hypotheses of Theorem 2? This function is decreasing for $x > 0$ (so it is monotonic for $x > 0$). As before, $1/\sqrt{x+1} \sim 1/\sqrt{x}$. It is also true, as before, that $\int_1^n \mathrm{d}x/\sqrt{x} \to \infty$, but this time we actually need to check this by doing the calculation, since $f(x)$ itself is not diverging (which is what gave an easy reason to know the integral was diverging in the previous example):

$$\int_1^n \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{n} - 2 \sim 2\sqrt{n}.$$

This diverges with n, so Theorem 2 tells us that

$$\sum_{k=1}^n \frac{1}{\sqrt{k}} \sim \int_1^n \frac{\mathrm{d}x}{\sqrt{x}} \sim 2\sqrt{n}.$$

The bottom line is this: if you want to estimate the asymptotic growth of $\sum_{k=1}^{n} f(k)$, provided $f(x)$ is either only increasing or only decreasing for large x, and provided $f(x) \sim f(x+1)$, take a look at the integral $\int_1^n f(x)\,\mathrm{d}x$. If the integral diverges, it provides an asymptotic estimate on the sum. (If the integral converges, then the sum converges too, but there is no reason for the values to be the same; consider $\sum_{k\geq 1} 1/k^2 = \pi^2/6$ and $\int_1^\infty \mathrm{d}x/x^2 = 1$.)

Now we prove Theorem 2. After the proof we will give another application to sum estimation.

Proof. We first assume, for large x, that $f(x)$ is increasing. Then, when the integer n is large, we have $f(n) \leq f(x) \leq f(n+1)$ for $n \leq x \leq n+1$, so

$$f(n) \leq \int_n^{n+1} f(x)\,\mathrm{d}x \leq f(n+1). \tag{2}$$

Dividing by $f(n)$,

$$1 \leq \frac{\int_n^{n+1} f(x)\,\mathrm{d}x}{f(n)} \leq \frac{f(n+1)}{f(n)}$$

for large integers n. Since, by hypothesis, $f(n) \sim f(n+1)$, the right side tends to 1 as $n \to \infty$. Thus (by the squeeze theorem) the ratio in the middle tends to 1, which means the numerator and denominator are asymptotic:

$$f(n) \sim \int_n^{n+1} f(x)\,\mathrm{d}x. \tag{3}$$

If instead we have $f(x)$ decreasing for large x, then we replace (2) with

$$f(n+1) \leq \int_n^{n+1} f(x)\,\mathrm{d}x \leq f(n) \tag{4}$$

and divide by $f(n)$ again (and let $n \to \infty$) to see (3) is still true.

Now we feed (3) into Theorem 1:

$$\sum_{k=1}^{n} f(k) \sim \sum_{k=1}^{n} \int_k^{k+1} f(x)\,\mathrm{d}x = \int_1^{n+1} f(x)\,\mathrm{d}x, \tag{5}$$

provided either of the two sums diverges as $n \to \infty$. Well, one of our hypotheses was that $\int_1^n f(x)\,\mathrm{d}x \to \infty$ as $n \to \infty$, so the second partial sum above diverges with n (just replace n with $n+1$). Thus, the application of Theorem 1 to derive (5) is correct.

Comparing (5) with the conclusion of Theorem 2, the last thing we need to do is check that, insofar as asymptotic estimates are concerned, we can replace the integral from 1 to $n+1$ in (5) with the integral from 1 to n:

$$\int_1^{n+1} f(x)\,\mathrm{d}x \sim \int_1^n f(x)\,\mathrm{d}x.$$

Both sides diverge with n (that was one of our hypotheses), so we can check their ratio tends to 1 by using L'Hopital's Rule:

$$\begin{aligned} \lim_{t\to\infty} \frac{\int_1^{t+1} f(x)\,\mathrm{d}x}{\int_1^t f(x)\,\mathrm{d}x} &= \lim_{t\to\infty} \frac{f(t+1)}{f(t)} \\ &= 1. \end{aligned}$$

Thus, in (5) we can replace $n+1$ with n in the integral, and that concludes the proof. $\square$

Example 4. As a third application of Theorem 2 we asymptotically estimate $\sum_{k=1}^n a^k/k$ where $a > 1$ is a fixed number. If the sum were $\sum_{k=1}^n a^k$, then we could sum the geometric series explicitly and find

$$\sum_{k=1}^n a^k = \frac{a^{n+1}-a}{a-1} \sim \frac{a^{n+1}}{a-1} = \frac{a}{a-1}a^n.$$

What happens when we sum a^k/k instead of a^k? It turns out that we just have to replace a^n in the final estimate with a^n/n, but the proof of this (via our above theorems) involves a curious detour through integrals and logarithms. We present the work in three steps.

Step 1: $\sum_{k=1}^n \frac{a^k}{k} \sim \frac{\log a}{a-1}\int_1^{n+1} \frac{a^u}{u}\,\mathrm{d}u$.
By L'Hôpital's Rule and the Fundamental Theorem of Calculus,

$$\begin{aligned}\lim_{t\to\infty} \frac{a^t/t}{\int_t^{t+1}(a^u/u)\,\mathrm{d}u} &= \lim_{t\to\infty} \frac{(ta^t\log a - a^t)/t^2}{a^{t+1}/(t+1) - a^t/t} \\ &= \lim_{t\to\infty} \frac{\log a - 1/t}{a\cdot\frac{t}{t+1} - 1} \\ &= \frac{\log a}{a-1}.\end{aligned}$$

Thus, $\frac{a^n}{n} \sim \frac{\log a}{a-1}\int_n^{n+1}\frac{a^u}{u}\,\mathrm{d}u$, so by Theorem 1 we're done.

Step 2: $\int_1^{n+1}\frac{a^u}{u}\,\mathrm{d}u \sim \frac{a}{\log a}\frac{a^n}{n}$.
By L'Hôpital's Rule and the Fundamental Theorem of Calculus,

$$\begin{aligned}\lim_{t\to\infty} \frac{\int_1^{t+1}(a^u/u)\,\mathrm{d}u}{a^t/t} &= \lim_{t\to\infty} \frac{a^{t+1}/(t+1)}{(ta^t\log a - a^t)/t^2} \\ &= \lim_{t\to\infty} \frac{a}{(\log a) - 1/t}\cdot\frac{t}{t+1} \\ &= \frac{a}{\log a}.\end{aligned}$$

Step 3: $\sum_{k=1}^n \frac{a^k}{k} \sim \frac{a}{a-1}\frac{a^n}{n}$.
This follows by feeding the result of Step 2 into the result of Step 1. The log terms cancel and we are done.

As an exercise for the reader, use Theorems 1 and 2 to derive the asymptotic estimate

$$\sum_{k=2}^n \frac{1}{\log k} \sim \int_2^n \frac{\mathrm{d}x}{\log x} \sim \frac{n}{\log n}.$$