ACCELERATING CONVERGENCE OF SERIES

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1. Introduction

An infinite series is the limit of its partial sums. However, it may take a large number of terms to get even a few correct digits for the series from its partial sums. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{1.1}$$

converges but the partial sums $s_N = 1 + 1/4 + 1/9 + \cdots + 1/N^2$ take a long time to settle down, as the table below illustrates, where s_N is truncated to 8 digits after the decimal point. The 1000th partial sum s_{1000} winds up matching the full series (1.1) only in 1.64.

N	10	20	25	50	100	1000
s_N	1.54976773	1.59616324	1.60572340	1.62513273	1.63498390	1.64393456

That the partial sums s_N converge slowly is related to the error bound from the integral test: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = s_N + r_N$ where

$$r_N = \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}. \tag{1.2}$$

To approximate (1.1) by s_N correctly to 3 digits after the decimal point means $r_N < .0001 = 1/10^4$, so the bound in (1.2) suggests we make $1/N \leq 1/10^4$, so $N \geq 10000$.[1] In the era before electronic computers, computing the 1000th partial sum of (1.1) was not feasible.

Our theme is speeding up convergence of a series $S = \sum_{n\geq 1} a_n$. This means rewriting S in a new way, say $S = \sum_{n\geq 1} a'_n$, so that the new tail $r'_N = \sum_{n>N} a'_n$ goes to 0 faster than the old tail $r_N = \sum_{n>N} a_n$. Such techniques are called series acceleration methods. For instance, we will accelerate (1.1) twice so the 20th accelerated partial sum s''_{20} is more accurate than the 1000th standard partial sum s_{1000} above.

2. Series with positive terms: Kummer's transformation

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a convergent series whose terms a_n are positive. If $\{b_n\}$ is a sequence growing at the same rate as $\{a_n\}$, meaning $\dfrac{a_n}{b_n} \to 1$ as $n \to \infty$, then $\displaystyle\sum_{n=1}^{\infty} b_n$ converges by the limit comparison test. If we happen to know the *exact* value of $B = \displaystyle\sum_{n=1}^{\infty} b_n$, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} (a_n - b_n) = B + \sum_{n=1}^{\infty} \left(1 - \frac{b_n}{a_n}\right) a_n \tag{2.1}$$

[1] The lower bound $r_N > \int_{N+1}^{\infty} dx/x^2 = 1/(N+1)$ proves $r_N < .0001 \Rightarrow N + 1 > 10000$, so $N \geq 10000$.

and the series on the right in (2.1) is likely to converge more rapidly than the series on the left since its terms tend to 0 more quickly than a_n on account of the new factor $1 - b_n/a_n$, which tends to 0. The identity (2.1) goes back to Kummer [5] and is called Kummer's transformation.

Example 2.1. We will use (2.1) to rewrite (1.1) as a new series where the remainder for the Nth partial sum decays faster than the error bound $1/N$ in (1.2).

A series whose terms grow at the same rate as (1.1) is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, which has exact value $B = 1$ from the simplest example of a telescoping series:

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{N} \to 1 \tag{2.2}$$

as $N \to \infty$. Taking $a_n = \dfrac{1}{n^2}$ and $b_n = \dfrac{1}{n(n+1)}$, so $\dfrac{b_n}{a_n} = \dfrac{n}{n+1}$, (2.1) says

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(1 - \frac{n}{n+1}\right) \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}. \tag{2.3}$$

Letting $s'_N = 1 + \displaystyle\sum_{n=1}^{N} \frac{1}{n^2(n+1)}$, here are its values (truncated to 8 digits after the decimal point) for the same N as in the previous table. This seems to converge faster than s_N.

N	10	20	25	50	100	1000
s'_N	1.64067682	1.64378229	1.64418494	1.64474057	1.64488489	1.64493356

We have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = s'_N + r'_N$ where $r'_N = \displaystyle\sum_{n=N+1}^{\infty} \frac{1}{n^2(n+1)}$. This tends to 0 faster than $\dfrac{1}{N}$:

$$r'_N < \sum_{n=N+1}^{\infty} \frac{1}{n^3} < \int_N^{\infty} \frac{dx}{x^3} = \frac{1}{2N^2}. \tag{2.4}$$

Therefore $r'_N < .0001$ if $1/(2N^2) \leq .0001$, which is equivalent to $N \geq 71$, and that's a great improvement on the bound $N \geq 10000$ to make $r_N < .0001$. Since $s'_{71} = 1.644837\ldots$, the series (1.1) lies between $s'_{71} - .0001 = 1.644737\ldots$ and $s'_{71} + .001 = 1.644937\ldots$, and since $1/(2N^2) = .0000005$ when $N = 1000$, the value of s'_{1000} tells us (1.1) is 1.64493 to 5 digits after the decimal point. By accelerating (1.1) even further we'll approximate it to 5 digits using a far earlier partial sum than the 1000-th.

From the series on the right in (2.3), let $a_n = \dfrac{1}{n^2(n+1)}$. A sequence that grows at the same rate as a_n is $b_n = \dfrac{1}{n(n+1)(n+2)}$, and we can compute $\displaystyle\sum_{n=1}^{\infty} b_n$ exactly using a telescoping series: as $N \to \infty$,

$$\sum_{n=1}^{N} b_n = \sum_{n=1}^{N} \left(\frac{1/2}{n(n+1)} - \frac{1/2}{(n+1)(n+2)}\right) = \frac{1}{4} - \frac{1/2}{(N+1)(N+2)} \to \frac{1}{4}. \tag{2.5}$$

In (2.1) with $a_n = \dfrac{1}{n^2(n+1)}$ and $b_n = \dfrac{1}{n(n+1)(n+2)}$, we have $B = \dfrac{1}{4}$ and $\dfrac{b_n}{a_n} = \dfrac{n}{n+2}$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \frac{1}{4} + \sum_{n=1}^{\infty} \left(1 - \frac{n}{n+2}\right) \frac{1}{n^2(n+1)} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}.$$

Feeding this into the right side of (2.3),

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}. \tag{2.6}$$

When $s_N'' = 1 + \dfrac{1}{4} + \displaystyle\sum_{n=1}^{N} \frac{2}{n^2(n+1)(n+2)}$, the next table exhibits faster convergence than previous tables for s_N and s_N'' for the same values of N.

N	10	20	25	50	100	1000
s_N''	1.64446470	1.64486454	1.64489719	1.64492911	1.64493342	1.64493406

Letting $r_N'' = \displaystyle\sum_{n=N+1}^{\infty} \frac{2}{n^2(n+1)(n+2)}$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = s_N'' + r_N''$, we have

$$r_N'' < \sum_{n=N+1}^{\infty} \frac{2}{n^4} < \int_N^{\infty} \frac{2}{x^4}\, dx = \frac{2}{3N^3},$$

which improves on (2.4) by an extra power of N just as (2.4) improved on (1.2) by an extra power of N. We have $r_N'' < .0001$ if $2/(3N^3) < .0001$, which is equivalent to $N \geq 19$, so from the value of s_{20}'' in the table above, (1.1) is between $s_{20}'' - .0001 = 1.64476\ldots$ and $s_{20}'' + .0001 = 1.644964\ldots$: the series (1.1) is 1.644 to 3 digits after the decimal point.

Let's accelerate the series on the right in (2.6): for $a_n = \dfrac{2}{n^2(n+1)(n+2)}$, a sequence growing at the same rate that is exactly summable is $b_n = \dfrac{2}{n(n+1)(n+2)(n+3)}$, where

$$B = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{2/3}{n(n+1)(n+2)} - \frac{2/3}{(n+1)(n+2)(n+3)} \right) = \frac{1}{9} \tag{2.7}$$

and $\dfrac{b_n}{a_n} = \dfrac{n}{n+3}$, so (2.1) tells us

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)} &= \frac{1}{9} + \sum_{n=1}^{\infty} \left(1 - \frac{n}{n+3}\right) \frac{2}{n^2(n+1)(n+2)} \\ &= \frac{1}{9} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}. \end{aligned}$$

Feeding this into (2.6),

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}. \tag{2.8}$$

Setting $s_N''' = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \displaystyle\sum_{n=1}^{N} \frac{6}{n^2(n+1)(n+2)(n+3)}$, we have the following values.

N	10	20	25	50	100	1000
s_N'''	1.64485320	1.64492728	1.64493110	1.64493385	1.64493405	1.64493406

We have $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = s_N''' + r_N'''$ where $\displaystyle r_N''' = \sum_{n=N+1}^{\infty}\frac{6}{n^2(n+1)(n+2)(n+3)}$ has the bound

$$r_N''' < \sum_{n=N+1}^{\infty}\frac{6}{n^5} < \int_N^{\infty}\frac{6}{x^5}\,dx = \frac{6}{4N^4} = \frac{3}{2N^4},$$

so $r_{25}''' < .00000384$. Using the table above, (1.1) is between $s_{25}''' - .00000384 > 1.64492726$ and $s_{25}''' + .00000384 < 1.64493495$, so (1.1) is 1.6449 to 4 digits after the decimal point.

We can continue this process. For each $k \geq 1$, telescoping series like (2.2), (2.5), and (2.7) generalize to

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+k)} &= \sum_{n=1}^{\infty}\left(\frac{1/k}{n(n+1)\cdots(n+k-1)} - \frac{1/k}{(n+1)(n+2)\cdots(n+k)}\right)\\ &= \frac{1}{k\cdot k!} \qquad (2.9)\end{aligned}$$

and this lets us generalize (2.3), (2.6), and (2.8) to

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{j=1}^{k}\frac{1}{j^2} + \sum_{n=1}^{\infty}\frac{k!}{n^2(n+1)(n+2)\cdots(n+k)} \qquad (2.10)$$

for each $k \geq 0$, where the first sum on the right is 0 at $k = 0$. The remainder term $r_N^{(k)}$ for the Nth partial sum of the rightmost series in (2.10) satisfies

$$r_N^{(k)} < \int_N^{\infty}\frac{k!}{x^{k+2}}\,dx = \frac{k!/(k+1)}{N^{k+1}}. \qquad (2.11)$$

Put $k = 5$ in (2.10) and let $\displaystyle s_N^{(5)} = \sum_{n=1}^{5}\frac{1}{n^2} + \sum_{n=1}^{N}\frac{120}{n^2(n+1)(n+2)(n+3)(n+4)(n+5)}$.
We get the following values.

N	10	20	25	50	100	1000
$s_N^{(5)}$	1.64492895	1.64493391	1.64493402	1.64493406	1.64493406	1.64493406

By (2.11), $|\sum_{n=1}^{\infty}1/n^2 - s_{20}^{(5)}| = |r_{20}^{(5)}| < (120/6)/20^6 = 1/20^5 = .0000003125$, which puts (1.1) between $s_{20}^{(5)} - .0000003125 = 1.6449335\ldots$ and $s_{20}^{(5)} + .0000003125 = 1.6449342\ldots$.

The series (1.1) that we have been finding good approximations to has an exact formula: $\dfrac{\pi^2}{6} = 1.6449340\ldots$. This beautiful and unexpected result was discovered by Euler in 1735, when he was still in his 20s, and it is what first made him famous. Before finding the exact value $\pi^2/6$, Euler created an acceleration method in 1731 to estimate (1.1) to 6 digits after the decimal point, which was far beyond feasible hand calculations using the terms in (1.1). (Figure 1 shows Euler's estimate on the second line, taken from the end of his article.) An account of this work is in [6], and the original paper (and an English translation) is in [1].

480453; ergo ſumma ſeriei 1 + 1/4 + 1/9 + 1/16 +
etc. eſt = 1, 644934 q.p. Si quis autem huius

FIGURE 1. End of Euler's article where $\sum_{n\geq 1}1/n^2$ is estimated as 1.644934.

Example 2.2. Consider $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$. Unlike (1.1), there is *no* known formula for this series in terms of more familiar numbers. We will estimate the series by accelerating it four times.

The nth term $a_n=\dfrac{1}{n^3}$ grows at the same rate as $b_n=\dfrac{1}{n(n+1)(n+2)}$, and we know the exact value of $\displaystyle\sum_{n=1}^{\infty}b_n$: by (2.5), it is $\dfrac{1}{4}$, so by (2.1)

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{1}{4}+\sum_{n=1}^{\infty}\left(1-\frac{b_n}{a_n}\right)a_n=\frac{1}{4}+\sum_{n=1}^{\infty}\frac{3n+2}{n^3(n+1)(n+2)}.$$

Now let $a_n=\dfrac{3n+2}{n^3(n+1)(n+2)}$, so a_n grows like $\dfrac{3}{n^4}$. A sequence growing at the same rate whose exact sum is known is $b_n=\dfrac{3}{n(n+1)(n+2)(n+3)}$: by (2.9),

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{3}{n(n+1)(n+2)(n+3)}=\frac{3}{3\cdot 3!}=\frac{1}{6},$$

so by (2.1) and algebra

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{1}{4}+\frac{1}{6}+\sum_{n=1}^{\infty}\left(1-\frac{b_n}{a_n}\right)a_n=\frac{1}{4}+\frac{1}{6}+\sum_{n=1}^{\infty}\frac{11n+6}{n^3(n+1)(n+2)(n+3)}.$$

Next let $a_n=\dfrac{11n+6}{n^3(n+1)(n+2)(n+3)}$, which grows like $\dfrac{11}{n^5}$. A sequence growing at the same rate whose exact sum is known is $b_n=\dfrac{11}{n(n+1)(n+2)(n+3)(n+4)}$: by (2.9),

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{11}{n(n+1)(n+2)(n+3)(n+4)}=\frac{11}{4\cdot 4!}=\frac{11}{96},$$

so by (2.1) and algebra

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{1}{4}+\frac{1}{6}+\frac{11}{96}+\sum_{n=1}^{\infty}\left(1-\frac{b_n}{a_n}\right)a_n=\frac{1}{4}+\frac{1}{6}+\frac{11}{96}+\sum_{n=1}^{\infty}\frac{50n+24}{n^3(n+1)\cdots(n+4)}.$$

It is left to the reader to derive the next acceleration, which is

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{1}{4}+\frac{1}{6}+\frac{11}{96}+\frac{1}{12}+\sum_{n=1}^{\infty}\frac{274n+120}{n^3(n+1)\cdots(n+5)}.$$

We now have five partial sums that each tend to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ as $N\to\infty$:

$$s_N=\sum_{n=1}^{N}\frac{1}{n^3},\quad s'_N=\frac{1}{4}+\sum_{n=1}^{N}\frac{3n+2}{n^3(n+1)(n+2)},\quad s''_N=\frac{1}{4}+\frac{1}{6}+\sum_{n=1}^{N}\frac{11n+6}{n^3(n+1)(n+2)(n+3)},$$

$$\begin{aligned}s'''_N&=\frac{1}{4}+\frac{1}{6}+\frac{11}{96}+\sum_{n=1}^{N}\frac{50n+24}{n^3(n+1)(n+2)(n+3)(n+4)},\\ s^{(4)}_N&=\frac{1}{4}+\frac{1}{6}+\frac{11}{96}+\frac{1}{12}+\sum_{n=1}^{N}\frac{274n+120}{n^3(n+1)(n+2)(n+3)(n+4)(n+5)}.\end{aligned}$$

The table below compares these partial sums for several values of N, each partial sum being truncated (not rounded) to 8 digits after the decimal point.

N	10	20	25	50	100	1000
s_N	1.19753198	1.20086784	1.20128826	1.20186086	1.20200740	1.20205640
s'_N	1.20131986	1.20195009	1.20200051	1.20204940	1.20205593	1.20205690
s''_N	1.20190261	1.20204420	1.20205138	1.20205651	1.20205687	1.20205690
s'''_N	1.20201708	1.20205498	1.20205621	1.20205687	1.20205690	1.20205690
$s_N^{(4)}$	1.20204483	1.20205655	1.20205679	1.20205690	1.20205690	1.20205690

We can bound the remainder term for each partial sum using the integral test, as in our previous example:

$$r_N := \sum_{n=N+1}^{\infty} \frac{1}{n^3} < \int_N^{\infty} \frac{dx}{x^3} = \frac{1}{2N^2},$$

$$r'_N := \sum_{n=N+1}^{\infty} \frac{3n+2}{n^3(n+1)(n+2)} < \sum_{n=N+1}^{\infty} \frac{3}{n^3(n+2)} < \int_N \frac{3}{x^4}\,dx = \frac{1}{N^3},$$

$$r''_N := \sum_{n=N+1}^{\infty} \frac{11n+6}{n^3(n+1)(n+2)(n+3)} < \sum_{n=N+1}^{\infty} \frac{11}{n^3(n+2)(n+3)} < \int_N \frac{11}{x^5}\,dx = \frac{11}{4N^4},$$

$$r'''_N := \sum_{n=N+1}^{\infty} \frac{50n+24}{n^3(n+1)(n+2)(n+3)(n+4)} < \sum_{n=N+1}^{\infty} \frac{50}{n^6} < \int_N \frac{50}{x^6}\,dx = \frac{50}{5N^5} = \frac{10}{N^5},$$

and

$$r_N^{(4)} := \sum_{n=N+1}^{\infty} \frac{274n+120}{n^3(n+1)(n+2)\cdots(n+5)} < \sum_{n=N+1}^{\infty} \frac{274}{n^7} < \int_N \frac{274}{x^7}\,dx = \frac{274}{6N^6} = \frac{137}{3N^6}.$$

These bounds imply $r_N < .00001$ for $N \geq 224$, $r'_N < .00001$ for $N \geq 47$, $r''_N < .00001$ for $N \geq 23$, $r'''_N < .00001$ for $N \geq 16$, and $r_N^{(4)} < .00001$ for $N \geq 13$. Using the bounds on r'''_N, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ lies between $s'''_{20} - .00001 = 1.202044\ldots$ and $s'''_{20} + .00001 = 1.202064\ldots$. We also have $r_N^{(4)} < .000001$ for $N \geq 19$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ lies between $s_{20}^{(4)} - .000001 = 1.2020555\ldots$ and $s_{20}^{(4)} + .000001 = 1.2020575\ldots$.

Analogous to the k-fold acceleration (2.10) for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a k-fold acceleration of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \sum_{j=1}^{k} \frac{c_j}{(j+1)(j+1)!} + \sum_{n=1}^{\infty} \frac{c_{k+1}n + (k+1)!}{n^3(n+1)\cdots(n+k+1)}$$

for each $k \geq 0$, where the first sum on the right is 0 at $k = 0$ and $c_k = 1, 3, 11, 50, 274, \ldots$ is determined by the recursive relation $c_1 = 1$ and $c_k = kc_{k-1} + (k-1)!$ for $k \geq 2$. (The integers c_k are the unsigned Stirling numbers of the first kind that count the number of permutations of the set $\{1, \ldots, k+1\}$ having 2 disjoint cycles.)

3. Alternating series: Euler's transformation

The Leibniz series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$

which equivalently says

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \frac{4}{17} - \frac{4}{19} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}, \tag{3.1}$$

converges very slowly. For example, the 100th partial sum of the series in (3.1) is $3.151\ldots$, which is accurate to only one digit past the decimal point.

We will describe a method due to Euler for accelerating the convergence of alternating series[2], and illustrate it for both (3.1) and the alternating harmonic series

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}. \tag{3.2}$$

Euler's basic idea is that a convergent alternating series

$$S = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots \tag{3.3}$$

can be rewritten as

$$S = \frac{a_0}{2} + \left(\frac{a_0}{2} - \frac{a_1}{2}\right) - \left(\frac{a_1}{2} - \frac{a_2}{2}\right) + \left(\frac{a_2}{2} - \frac{a_3}{2}\right) - \left(\frac{a_3}{2} - \frac{a_4}{2}\right) + \left(\frac{a_4}{2} - \frac{a_5}{2}\right) - \cdots, \tag{3.4}$$

where each term of the original series is split in half and combined with half of the adjacent terms on both sides of the original series (except the first term a_0, where a single $a_0/2$ is left on its own). The *order* of addition has not changed in passing from (3.3) to (3.4), so the value of the series does not change. Since the terms $a_n/2 - a_{n+1}/2 = (a_n - a_{n+1})/2$ may have a faster decay rate than the original terms a_n, applying this transformation multiple times can accelerate the convergence in an impressive way.

Example 3.1. Applying (3.3) $\rightsquigarrow$ (3.4) to (3.1) turns this series into

$$\begin{aligned}\pi &= 2 + \left(2 - \frac{2}{3}\right) - \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{7}\right) - \left(\frac{2}{7} - \frac{2}{9}\right) + \left(\frac{2}{9} - \frac{2}{11}\right) - \left(\frac{2}{11} - \frac{2}{13}\right) + \cdots \\ &= 2 + \frac{4}{1\cdot 3} - \frac{4}{3\cdot 5} + \frac{4}{5\cdot 7} - \frac{4}{7\cdot 9} + \frac{4}{9\cdot 11} - \frac{4}{11\cdot 13} + \cdots.\end{aligned}$$

We have changed (3.1) into

$$\pi = 2 + \sum_{n=0}^{\infty} \frac{(-1)^n 4}{(2n+1)(2n+3)} \tag{3.5}$$

by replacing $a_n = \dfrac{4}{2n+1}$ in (3.3) with

$$a_n' = \frac{a_n}{2} - \frac{a_{n+1}}{2} = \frac{2}{2n+1} - \frac{2}{2n+3} = \frac{2(2n+3) - 2(2n+1)}{(2n+1)(2n+3)} = \frac{4}{(2n+1)(2n+3)}. \tag{3.6}$$

[2]Euler gives a brief account of accelerating the series in our Examples 3.1 and 3.2 in [2, Chap. 1, Part II] (pp. 236-237 of the original Latin and pp. 399-400 in the English translation). See also [4, Sect. 35B].

Now view the alternating series in (3.5) as an instance of (3.3) and transform it using (3.4):

$$\begin{aligned}\pi &= 2 + \left(\frac{4}{1\cdot 3} - \frac{4}{3\cdot 5} + \frac{4}{5\cdot 7} - \frac{4}{7\cdot 9} + \frac{4}{9\cdot 11} - \frac{4}{11\cdot 13} + \cdots\right)\\ &\overset{(3.4)}{=} 2 + \frac{2}{1\cdot 3} + \left(\frac{2}{1\cdot 3} - \frac{2}{3\cdot 5}\right) - \left(\frac{2}{3\cdot 5} - \frac{2}{5\cdot 7}\right) + \left(\frac{2}{5\cdot 7} - \frac{2}{7\cdot 9}\right) - \left(\frac{2}{7\cdot 9} - \frac{2}{9\cdot 11}\right) + \cdots\\ &= 2 + \frac{2}{3} + \frac{8}{1\cdot 3\cdot 5} - \frac{8}{3\cdot 5\cdot 7} + \frac{8}{5\cdot 7\cdot 9} - \frac{8}{7\cdot 9\cdot 11} + \cdots,\end{aligned}$$

which has changed (3.5) into

$$\pi = 2 + \frac{2}{3} + \sum_{n=0}^{\infty} \frac{(-1)^n 8}{(2n+1)(2n+3)(2n+5)} \tag{3.7}$$

by replacing a'_n in (3.6) with

$$a''_n = \frac{a'_n}{2} - \frac{a'_{n+1}}{2} = \frac{4}{(2n+1)(2n+3)} - \frac{4}{(2n+3)(2n+5)} = \frac{8}{(2n+1)(2n+3)(2n+5)}. \tag{3.8}$$

Next view the alternating series in (3.7) as (3.3) and transform it using (3.4):

$$\begin{aligned}\pi &= 2 + \frac{2}{3} + \left(\frac{8}{1\cdot 3\cdot 5} - \frac{8}{3\cdot 5\cdot 7} + \frac{8}{5\cdot 7\cdot 9} - \frac{8}{7\cdot 9\cdot 11} + \cdots\right)\\ &\overset{(3.4)}{=} 2 + \frac{2}{3} + \frac{4}{15} + \left(\frac{4}{1\cdot 3\cdot 5} - \frac{4}{3\cdot 5\cdot 7}\right) - \left(\frac{4}{3\cdot 5\cdot 7} - \frac{4}{5\cdot 7\cdot 11}\right) + \cdots\\ &= 2 + \frac{2}{3} + \frac{4}{15} + \frac{24}{1\cdot 3\cdot 5\cdot 7} - \frac{24}{3\cdot 5\cdot 7\cdot 9} + \cdots.\end{aligned}$$

We have changed (3.7) into

$$\pi = 2 + \frac{2}{3} + \frac{4}{15} + \sum_{n=0}^{\infty} \frac{(-1)^n 24}{(2n+1)(2n+3)(2n+5)(2n+7)} \tag{3.9}$$

by replacing a''_n in (3.8) with

$$a'''_n = \frac{a''_n}{2} - \frac{a''_{n+1}}{2} = \frac{24}{(2n+1)(2n+3)(2n+5)(2n+7)}. \tag{3.10}$$

Applying this process two more times, we get

$$\pi = 2 + \frac{2}{3} + \frac{4}{15} + \frac{12}{105} + \sum_{n=0}^{\infty} \frac{(-1)^n 96}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)} \tag{3.11}$$

and

$$\pi = 2 + \frac{2}{3} + \frac{4}{15} + \frac{12}{105} + \frac{48}{945} + \sum_{n=0}^{\infty} \frac{(-1)^n 480}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)}. \tag{3.12}$$

We now have six partial sums that each tend to π as $N \to \infty$:

$$\begin{aligned}
s_N &= \sum_{n=0}^{N} \frac{(-1)^n 4}{2n+1}, \\
s'_N &= 2 + \sum_{n=0}^{N} \frac{(-1)^n 4}{(2n+1)(2n+3)}, \\
s''_N &= 2 + \frac{2}{3} + \sum_{n=0}^{N} \frac{(-1)^n 8}{(2n+1)(2n+3)(2n+5)}, \\
s'''_N &= 2 + \frac{2}{3} + \frac{4}{15} + \sum_{n=0}^{N} \frac{(-1)^n 24}{(2n+1)(2n+3)(2n+5)(2n+7)}, \\
s^{(4)}_N &= 2 + \frac{2}{3} + \frac{4}{15} + \frac{12}{105} + \sum_{n=0}^{N} \frac{(-1)^n 96}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)}, \\
s^{(5)}_N &= 2 + \frac{2}{3} + \frac{4}{15} + \frac{12}{105} + \frac{48}{945} + \sum_{n=0}^{N} \frac{(-1)^n 480}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)}.
\end{aligned}$$

The table below lists these partial sums at $N = 10$, 20, 25, 50, 100, and 1000 truncated (not rounded) to 8 digits after the decimal point. While s_{10} is only accurate to one digit, $s^{(5)}_{10}$ is accurate to 6 digits. While s_{100} is only accurate to two digits, $s^{(5)}_{100}$ is accurate to 11 digits (the 9th and 10th digits after the decimal point are not in the table).

N	10	20	25	50	100	1000
s_N	3.23231580	3.18918478	3.10314531	3.16119861	3.15149340	3.14259165
s'_N	3.14535928	3.14267315	3.14088116	3.14178113	3.14164118	3.14159315
s''_N	3.14188102	3.14163956	3.14156726	3.14159620	3.14159312	3.14159265
s'''_N	3.14162337	3.14159558	3.14159134	3.14159275	3.14159266	3.14159265
$s^{(4)}_N$	3.14159672	3.14159288	3.14159256	3.14159265	3.14159265	3.14159265
$s^{(5)}_N$	3.14159328	3.14159267	3.14159264	3.14159265	3.14159265	3.14159265

The reason accelerated series tend to converge faster is that their terms decay to 0 at ever faster rates. Terms in the successive series for π – (3.1), (3.5), (3.7), (3.9), (3.11), and (3.12) – decay as follows:

$$\begin{aligned}
a_n &= \frac{4}{2n+1} \sim \frac{2}{n}, \\
a'_n &= \frac{4}{(2n+1)(2n+3)} \sim \frac{1}{n^2}, \\
a''_n &= \frac{8}{(2n+1)(2n+3)(2n+5)} \sim \frac{1}{n^3}, \\
a'''_n &= \frac{24}{(2n+1)(2n+3)(2n+5)(2n+7)} \sim \frac{3}{2n^4}, \\
a^{(4)}_n &= \frac{96}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)} \sim \frac{3}{n^5}, \\
a^{(5)}_n &= \frac{480}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)} \sim \frac{15}{2n^6}.
\end{aligned}$$

In general, after applying k series accelerations to (3.1) we have

$$\pi = \sum_{j=0}^{k-1} \frac{2(j!)}{1\cdot 3\cdot 5\cdots(2j+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n 4(k!)}{(2n+1)(2n+3)\cdots(2n+2k+1)}, \tag{3.13}$$

where the first sum in (3.13) is 0 for $k = 0$. This formula at $k = 0$, 1, 2, 3, 4, and 5 is (3.1), (3.5), (3.7), (3.9), (3.11), and (3.12) respectively, and for each k the magnitude of the nth term in the second series in (3.13) decays like $1/n^{k+1}$ up to a scaling factor: as $n \to \infty$,

$$\frac{4(k!)}{(2n+1)(2n+3)\cdots(2n+2k+1)} \sim \frac{4(k!)}{(2n)^{k+1}} = \frac{k!/2^{k-1}}{n^{k+1}}.$$

In an answer on https://math.stackexchange.com/questions/1702694/why-is-the-leibniz-method-for-approximating-pi-so-inefficient the series (3.1) is accelerated 24 times.

Error bounds on the remainder for each series for π can be obtained from the alternating series test: the absolute value of the first omitted term is a bound. Writing $r_N^{(i)} = \pi - s_N^{(i)}$,

$$|r_N| < \frac{4}{2(N+1)} < \frac{2}{N}, \quad |r_N'| < \frac{4}{4N^2} = \frac{1}{N^2}, \quad |r_N''| < \frac{8}{8N^3} = \frac{1}{N^3},$$
$$|r_N'''| < \frac{24}{16N^4} = \frac{3}{2N^4}, \quad |r_N^{(4)}| < \frac{96}{32N^5} = \frac{3}{N^5}, \quad |r_N^{(5)}| < \frac{480}{64N^6} = \frac{15}{2N^6}.$$

For example, $|r_N^{(4)}| < .000001$ if $3/N^5 < .000001$, which is the same as $N \geq 20$. Thus π is between $s_{20}^{(4)} - .000001 = 3.141591\ldots$ and $s_{20}^{(4)} + .000001 = 3.141593\ldots$.

Example 3.2. Now we turn to the alternating harmonic series (3.2), and will be more brief than we were with the series for π. Write (3.2) as $a_1 - a_2 + a_3 - a_4 + \cdots = \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \dfrac{1}{n}$. Accelerating (3.2) once turns that series into

$$\frac{a_1}{2} + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{a_n}{2} - \frac{a_{n+1}}{2}\right) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(n+1)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{12} + \frac{1}{24} - \frac{1}{40} + \cdots. \tag{3.14}$$

Accelerating (3.14) makes it

$$\frac{1}{2} + \frac{1}{8} + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{4n(n+1)} - \frac{1}{4(n+1)(n+2)}\right) = \frac{1}{2} + \frac{1}{8} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(n+1)(n+2)} \tag{3.15}$$

and the reader should check as an exercise that the next few accelerations of (3.2) are

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3}{4n(n+1)(n+2)(n+3)}, \tag{3.16}$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3}{2n(n+1)(n+2)(n+3)(n+4)}, \tag{3.17}$$

and

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}15}{4n(n+1)(n+2)(n+3)(n+4)(n+5)}. \tag{3.18}$$

In the table below we list partial sums of (3.2) and its accelerated forms (3.14), (3.15), (3.16), (3.17), and (3.18). The notation $s_N^{(i)}$ in the first column is, by analogy with the series for π, the ith accelerated form of (3.2), for $0 \leq i \leq 5$, with the sum running up to $n = N$.

N	10	20	25	50	100	1000
s_N	.64563492	.66877140	.71274749	.68324716	.68817217	.69264743
s'_N	.69108946	.69258092	.69351673	.69305108	.69312267	.69314693
s''_N	.69298340	.69312205	.69316060	.69314535	.69314694	.69314718
s'''_N	.69312909	.69314557	.69314788	.69314712	.69314717	.69314718
$s_N^{(4)}$	.69314470	.69314705	.69314722	.69314717	.69314718	.69314718
$s_N^{(5)}$	.69314678	.69314716	.69314718	.69314718	.69314718	.69314718

Since all these series fit the alternating series test, we can bound remainders using the first missing term. For example, from (3.18), $|r_{20}^{(5)}| \leq 15/(4(21^6)) \approx 4.37/10^8$, so from the value of $s_{20}^{(5)}$ we know (3.2) equals .693147.... By comparison, the 1000th partial sum of (3.2) is accurate to just two digits!

Generalizing the series (3.14)–(3.18), after k accelerations

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{j=1}^{k} \frac{1}{2^j j} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} k!}{2^k n(n+1)\cdots(n+k)}, \tag{3.19}$$

where the first sum on the right in (3.19) is 0 when $k = 0$, and for each k the nth term of the second series on the right in (3.19) decays like $1/n^{k+1}$ up to a scaling factor: as $n \to \infty$,

$$\frac{k!}{2^k n(n+1)\cdots(n+k)} \sim \frac{k!/2^k}{n^{k+1}}.$$

We can describe the effect of applying Euler's transformation k times to $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n$ by using notation from the difference calculus. For a sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)$, its *first discrete difference* is the sequence $\Delta\mathbf{a} = (a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots)$, so $(\Delta\mathbf{a})(n) = a_{n+1} - a_n$. The second discrete difference of $\mathbf{a}$ is $\Delta^2\mathbf{a} = \Delta(\Delta\mathbf{a})$, which starts out as

$$((\Delta\mathbf{a})(1) - (\Delta\mathbf{a})(0), (\Delta\mathbf{a})(2) - (\Delta\mathbf{a})(1), \ldots) = (a_2 - 2a_1 + a_0, a_3 - 2a_2 + a_1, \ldots)$$

and in general the kth discrete difference of $\mathbf{a}$ is $\Delta^k\mathbf{a} = \Delta(\Delta^{k-1}\mathbf{a})$. The formula $(\Delta^2\mathbf{a})(n) = a_{n+2} - 2a_{n+1} + a_n$ suggests a connection with binomial coefficients using alternating signs. Indeed, for $k \geq 0$ we have

$$(\Delta^k\mathbf{a})(n) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_{n+j}$$

for each $n \geq 0$. In particular, $(\Delta^k\mathbf{a})(0) = \displaystyle\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_j = a_k - ka_{k-1} + \cdots + (-1)^k a_0$.

With this notation, Euler's transformation in (3.4) consists of rewriting $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n$ as

$$\frac{a_0}{2} + \sum_{n=0}^{\infty}(-1)^n \frac{a_n - a_{n+1}}{2} = \frac{a_0}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(a_{n+1} - a_n) = \frac{a_0}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\Delta\mathbf{a})(n). \tag{3.20}$$

Apply Euler's transformation to the series on the right in (3.20) gives us

$$\sum_{n=0}^{\infty}(-1)^n(\Delta\mathbf{a})(n) = \frac{(\Delta\mathbf{a})(0)}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\Delta(\Delta\mathbf{a}))(n) = \frac{a_1 - a_0}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\Delta^2\mathbf{a})(n),$$

and feeding this into (3.20) shows $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n$ is

$$\frac{a_0}{2} - \frac{1}{2}\left(\frac{a_1 - a_0}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\Delta^2\mathbf{a})(n)\right) = \frac{a_0}{2} - \frac{(\Delta\mathbf{a})(1)}{4} + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(\Delta^2\mathbf{a})(n).$$

In general, applying Euler's transformation k times leads to the acceleration formula

$$\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{j=0}^{k-1}\frac{(-1)^j}{2^{j+1}}(\Delta^j\mathbf{a})(0) + \left(\frac{-1}{2}\right)^k\sum_{n=0}^{\infty}(-1)^n(\Delta^k\mathbf{a})(n), \tag{3.21}$$

where the first (finite) series on the right is 0 at $k = 0$.

Remark 3.3. While we are interested in examples where Euler's transformation speeds up convergence, it does not always have such an effect. For example, if $a_n = r^n$ with $|r| < 1$ then $(\Delta\mathbf{a})(n) = r^{n+1} - r^n = (r-1)r^n = (r-1)a_n$, so Euler's transformation on a geometric series leads to no improvement:

$$\sum_{n=0}^{\infty}(-1)^n r^n = \frac{1}{2} - \frac{r-1}{2}\sum_{n=0}^{\infty}(-1)^n r^n.$$

A version of Euler's transformation can be applied to any convergent series that can be written as a power series $\displaystyle\sum_{n=0}^{\infty} a_n c^n$ for some c in $[-1, 1)$, not just for alternating series (the case $c = -1$):

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n c^n &= a_0 + a_1 c + a_2 c^2 + a_3 c^3 + \cdots \\
&= a_0\frac{1-c}{1-c} + a_1\frac{c-c^2}{1-c} + a_2\frac{c^2-c^3}{1-c} + a_3\frac{c^3-c^4}{1-c} + \cdots \\
&= \frac{a_0}{1-c} + \frac{-a_0+a_1}{1-c}c + \frac{-a_1+a_2}{1-c}c^2 + \frac{-a_2+a_3}{1-c}c^3 + \cdots \\
&= \frac{a_0}{1-c} + \frac{a_1-a_0}{1-c}c + \frac{a_2-a_1}{1-c}c^2 + \frac{a_3-a_2}{1-c}c^3 + \cdots \\
&= \frac{a_0}{1-c} + \frac{c}{1-c}\sum_{n=0}^{\infty}(a_{n+1}-a_n)c^n \\
&= \frac{a_0}{1-c} + \frac{c}{1-c}\sum_{n=0}^{\infty}(\Delta\mathbf{a})(n)c^n.
\end{aligned}$$

When $c = -1$ this is (3.20), and in case the result seems like a trick it could also be derived using summation by parts with $u_n = a_n$ and $v_n = c^{n+1}/(c-1)$. Repeating this process k times for $k \geq 0$,

$$\sum_{n=0}^{\infty} a_n c^n = \sum_{j=0}^{k-1}\frac{c^j}{(1-c)^{j+1}}(\Delta^j\mathbf{a})(0) + \frac{c^k}{(1-c)^k}\sum_{n=0}^{\infty}(\Delta^k\mathbf{a})(n)c^n,$$

where the first sum on the right is 0 at $k = 0$. At $c = -1$ the above formula is (3.21).

For more on this, see [3] and [4, Sect. 33, 35], but watch out: in [4], $(\Delta\mathbf{a})(n) = a_n - a_{n+1}$. That is the negative of our convention, so Δ^n in [4] is our $(-1)^n\Delta^n$.

4. More speed-up methods

We briefly mention two further techniques for accelerating convergence of series.

(1) The *Shanks transformation* is applied to the partial sums of a series, not to the terms of the series. If $S = \sum_{n\geq 1} a_n$ has partial sums s_N for $N \geq 1$ then the Shanks transformation of the partial sums is the new sequence s'_N where

$$s'_N = s_{N+2} - \frac{(s_{N+2} - s_{N+1})^2}{s_{N+2} - 2s_{N+1} + s_N}$$

provided the denominators are not 0. For example, if $S = \ln 2 = .69314718\ldots = \sum_{n\geq 1}(-1)^{n-1}/n$ then $s_{100} = .688172\ldots$ is only accurate to one digit while $s'_{10} = .693065\ldots$ is accurate to 3 digits and $s'_{50} = .693146\ldots$ is accurate to 5 digits.

Check as an exercise that if $S = \sum_{n\geq 1} ar^{n-1}$ is a geometric series with $|r| < 1$ and $a \neq 0$, and $s_N = a + ar + \cdots + ar^{N-1}$ then $s'_N = a/(1-r)$ for all N. Thus the Shanks transformation accelerates every partial sum of geometric series directly to the full series in one step.

(2) The discrete Fourier transform of a sequence $a_0, a_1, \ldots, a_{m-1}$ in $\mathbf{C}$ is the new sequence $\widehat{a}_0, \widehat{a}_1, \ldots, \widehat{a}_{m-1}$ where $\widehat{a}_k = \sum_{j=0}^{m-1} a_k e^{-2\pi ijk/m}$. Calculating all of these (finite) series rapidly is an important task. To compute each $\widehat{a}_k$ from its definition requires m multiplications, so computing every $\widehat{a}_k$ requires $m \cdot m = m^2$ multiplications. The *fast Fourier transform* (FFT) is an alternate approach to computing the discrete Fourier transform that requires something on the order of at most $m \ln m$ operations, which is a big improvement on the naive approach directly from the definitions.

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