THE SPACE $c_0(K)$

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1. Introduction

Let $(K, |\cdot|)$ be a complete valued field. The space of sequences in K tending to 0,

$$c_0(K) = \{\mathbf{a} = (a_1, a_2, a_3, \ldots) : a_n \in K \text{ and } a_n \to 0 \text{ as } n \to \infty\},$$

is a basic example of a "sequence space" in analysis. Sequence spaces can be defined by conditions other than the terms tending to 0 (such as being a bounded sequence or having $\sum_{n\geq 1} |a_n|^2$ be convergent), but we will stick to the simple rule $a_n \to 0$ here. An example of an element in $c_0(K)$ is a sequence whose terms eventually equal 0 ($a_n = 0$ for all large n). When $|\cdot|$ is not the trivial absolute value on K, another example of an element of $c_0(K)$ is a power sequence $(1, x, x^2, x^3, \ldots)$ where $0 < |x| < 1$.[1] The termwise sum ($\mathbf{a}+\mathbf{b} = \{a_n + b_n\}$) and a scalar multiple ($\alpha\{a_n\} = \{\alpha a_n\}$) of a sequence in $c_0(K)$ are in $c_0(K)$, since if $a_n \to 0$ and $b_n \to 0$ then $a_n + b_n \to 0$ and $\alpha a_n \to 0$.

A natural way to measure the size a sequence $\mathbf{a}$ in $c_0(K)$ is by the maximum absolute value of the terms in $\mathbf{a}$:

$$||\mathbf{a}|| = \max_{n\geq 1} |a_n|.$$

Trivially $||\mathbf{a}|| \geq 0$, with equality if and only if $\mathbf{a} = \mathbf{0}$ (that is, $a_n = 0$ for all n) and $||\alpha\mathbf{a}|| = |\alpha|||\mathbf{a}||$ for $\mathbf{a} \in c_0(K)$ and $\alpha \in K$. In particular, $|| - \mathbf{a}|| = ||\mathbf{a}||$. There is also a triangle inequality: $||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$. Indeed, for all $n \geq 1$ we have $|a_n + b_n| \leq |a_n| + |b_n| \leq ||\mathbf{a}|| + ||\mathbf{b}||$, so taking the maximum over all n gives us $||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$. If $|\cdot|$ is non-archimedean then $||\cdot||$ is as well, *i.e.*, we have the strong triangle inequality $||\mathbf{a} + \mathbf{b}|| \leq \max(||\mathbf{a}||, ||\mathbf{b}||)$.

From the triangle inequality on $||\cdot||$ we get a metric on $c_0(K)$ by using the size of the difference:

$$d(\mathbf{a}, \mathbf{b}) = ||\mathbf{a} - \mathbf{b}||. \tag{1.1}$$

From now on it is understood that $c_0(K)$ is a metric space using (1.1).

We will show that $c_0(K)$ is a complete metric space, and when $K = \mathbf{R}$ and $K = \mathbf{Q}_p$ we will describe all the continuous linear functions $c_0(K) \to K$ ("generalized coordinates"). This will present a nice comparison between the way series behave in $\mathbf{R}$ and $\mathbf{Q}_p$.

2. Completeness of $c_0(K)$

Theorem 2.1. *The space $c_0(K)$ is complete for the metric (1.1).*

[1] If $|\cdot|$ is the trivial absolute value, so $|x| < 1 \Rightarrow x = 0$, then the only elements of $c_0(K)$ are the sequences in K whose terms eventually equal 0.

Proof. Let $\{\mathbf{a}_m\}$ be a Cauchy sequence in $c_0(K)$ (a "Cauchy sequence in a space of sequences"). To show this sequence has a limit in $c_0(K)$, first we will figure out what the limit should be, then we will show it lies in $c_0(K)$, and finally we will show it is a limit of the sequence.

Write $\mathbf{a}_m$ in component form as $(a_{m1}, a_{m2}, a_{m3}, \ldots)$, so $a_{mn} \in K$ and $\lim_{n\to\infty} a_{mn} = 0$ for each m. It is convenient to picture the sequences stacked on top of each other as

$$\begin{aligned} \mathbf{a}_1 &= (a_{11}, a_{12}, a_{13}, \ldots) \\ \mathbf{a}_2 &= (a_{21}, a_{22}, a_{23}, \ldots) \\ \mathbf{a}_3 &= (a_{31}, a_{32}, a_{33}, \ldots) \\ &\vdots \end{aligned}$$

since this suggests viewing the components all at once as an infinite matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where each row tends to 0. We will show limits can be taken along the columns to find a limit of the $\mathbf{a}_m$'s.

Since $\{\mathbf{a}_m\}$ is Cauchy in $c_0(K)$, for all $\varepsilon > 0$ there is an $M \geq 1$ such that $m, m' \geq M \Longrightarrow ||\mathbf{a}_m - \mathbf{a}_{m'}|| \leq \varepsilon$. That says

$$m, m' \geq M \Longrightarrow |a_{mn} - a_{m'n}| \leq \varepsilon \text{ for each } n \geq 1. \tag{2.1}$$

Therefore when n is fixed, the numbers $\{a_{mn}\}_{m\geq 1}$ are a Cauchy sequence in K, which is complete, so we have a limit: set $a_n = \lim_{m\to\infty} a_{mn}$. This is the limit of the nth column of the above matrix. We get a new sequence in K using the limits of each of the columns: set $\mathbf{a} = (a_1, a_2, a_3, \ldots)$. We want to show $\mathbf{a} \in c_0(K)$ and $||\mathbf{a} - \mathbf{a}_m|| \to 0$ as $m \to \infty$.

<u>$\mathbf{a}$ lies in $c_0(K)$</u>: For $\varepsilon > 0$ we want to find an $N \geq 1$ such that $n \geq N \Rightarrow |a_n| \leq \varepsilon$. From the Cauchy property of $\{\mathbf{a}_m\}$ in $c_0(K)$ there is an $M \geq 1$ such that (2.1) holds. Letting $m' \to \infty$ in (2.1) we get

$$m \geq M \Longrightarrow |a_{mn} - a_n| \leq \varepsilon \text{ for each } n \geq 1. \tag{2.2}$$

Using $m = M$ in (2.2), for each $n \geq 1$ we have

$$|a_n| = |a_n - a_{Mn} + a_{Mn}| \leq |a_n - a_{Mn}| + |a_{Mn}| \leq \varepsilon + |a_{Mn}|.$$

Since $\mathbf{a}_M \in c_0(K)$ we have $a_{Mn} \to 0$ as $n \to \infty$, so there is an $N \geq 1$ (depending on M) such that $n \geq N \Longrightarrow |a_{Mn}| \leq \varepsilon$. Therefore

$$n \geq N \Longrightarrow |a_n| \leq \varepsilon + |a_{Mn}| \leq 2\varepsilon. \tag{2.3}$$

Running through this argument with ε replaced by $\varepsilon/2$ so that we get $2(\varepsilon/2) = \varepsilon$ in (2.3), we have proved $a_n \to 0$ in K as $n \to \infty$. Thus $\mathbf{a} \in c_0(K)$.

<u>$\mathbf{a}$ is the limit of the $\mathbf{a}_m$'s</u>: For $\varepsilon > 0$ there is $M \geq 1$ making (2.2) hold. For all $m \geq M$,

$$||\mathbf{a}_m - \mathbf{a}|| = \max_{n\geq 1} |a_{mn} - a_n| \leq \varepsilon.$$

That proves the $\mathbf{a}_m$'s tend to $\mathbf{a}$ as $m \to \infty$. $\square$

Remark 2.2. The set of bounded sequences in K (that is, all $(a_1, a_2, \ldots)$ with $a_n \in K$ for which there's some $B > 0$ such that $|a_n| \leq B$ for all n) can be equipped with the same metric used on $c_0(K)$, except the role of a maximum on the absolute values has to be replaced by a supremum: $||\mathbf{a}|| = \sup_{n\geq 1} |a_n|$ and $d(\mathbf{a}, \mathbf{b}) = ||\mathbf{a} - \mathbf{b}|| = \sup_{n\geq 1} |a_n - b_n|$. This metric makes the set of all bounded sequences in K a complete metric space, and $c_0(K)$ is a closed subset of it. The proof of completeness is very similar to the proof of Theorem 2.1; the only difference is showing the supposed limit sequence is bounded instead of having its terms tend to 0, and the boundedness is slightly easier to establish. Details are left to the reader.

Corollary 2.3. *The set of sequences in $(K, |\cdot|)$ whose terms eventually equal 0, equipped with the metric (1.1), has completion $c_0(K)$.*

Proof. Every sequence in K whose terms eventually equal 0 lies in $c_0(K)$. Since we have proved $c_0(K)$ is complete, it is enough to show the sequences in K whose terms are eventually 0 are a dense subset of $c_0(K)$.

Fix a choice of $\mathbf{a} = (a_1, a_2, \ldots)$ in $c_0(K)$. Let $\mathbf{a}_n = (a_1, \ldots, a_n, 0, 0, \ldots)$ have the first n components from $\mathbf{a}$ and all later components equal to 0. Then

$$||\mathbf{a} - \mathbf{a}_n|| = ||(0, \ldots, 0, a_{n+1}, a_{n+2}, \ldots)|| = \max_{i>n} |a_i|.$$

By the definition of $\mathbf{a}$ belonging to $c_0(K)$, for each $\varepsilon > 0$ we have $|a_i| \leq \varepsilon$ for all large i, so there's some N making $|a_i| \leq \varepsilon$ for $i \geq N$. Thus $||\mathbf{a} - \mathbf{a}_n|| \leq \varepsilon$ for $n > N$, which proves the sequences in K with terms eventually equal to 0 are dense in $c_0(K)$. □

3. Continuous linear maps $c_0(\mathbf{R}) \to \mathbf{R}$ and $c_0(\mathbf{Q}_p) \to \mathbf{Q}_p$

For a field K, the standard coordinates on K^d are the functions $(a_1, \ldots, a_d) \mapsto a_i$ for $i = 1, \ldots, d$. As the vector varies, the ith coordinate can be regarded as a function $x^i \colon K^d \to K$ where $x^i((a_1, \ldots, a_d)) = a_i$. Each coordinate function is K-linear:

$$x^i(\mathbf{a} + \mathbf{a}') = x^i(\mathbf{a}) + x^i(\mathbf{a}'), \quad x^i(\alpha\mathbf{a}) = \alpha x^i(\mathbf{a})$$

for all $\mathbf{a}$ and $\mathbf{a}'$ in K^d and $\alpha \in K$. This is just saying that the i-th coordinate of a sum is the sum of the ith coordinates and the ith coordinate of a scalar multiple is the same scalar multiple of the ith coordinate, which is obvious from how vector addition and scalar multiplication in K^d are defined.

Each coordinate function $x^i \colon K^d \to K$ is a dot product with the ith standard basis vector: $x^i(\mathbf{a}) = \mathbf{a} \cdot e_i$, where e_i has 1 in the ith position and 0 elsewhere. More generally, for every $\mathbf{b}$ in K^d the function $K^d \to K$ defined by $\mathbf{a} \mapsto \mathbf{a} \cdot \mathbf{b}$ is K-linear. Taking the dot product on K^d with a fixed vector in K^d can be characterized by its linearity and having values in K:

Theorem 3.1. *The K-linear functions $\varphi \colon K^d \to K$ are functions of the form $\varphi(\mathbf{a}) = \mathbf{a} \cdot \mathbf{b}$ for some $\mathbf{b}$ in K^d, and each φ arises in this way for a unique $\mathbf{b}$ in K^d.*

Proof. For each $\mathbf{b} \in K^d$, it is easy to see that the function $\varphi_{\mathbf{b}} \colon K^d \to K$ defined by $\varphi_{\mathbf{b}}(\mathbf{a}) = \mathbf{a} \cdot \mathbf{b}$ is K-linear from algebraic properties of the dot product.

Conversely, assume $\varphi \colon K^d \to K$ is K-linear. Using linearity, we have

$$\varphi((a_1, \ldots, a_d)) = \varphi(a_1 e_1 + \cdots + a_d e_d) = a_1\varphi(e_1) + \cdots + a_d\varphi(e_d).$$

Letting $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (\varphi(e_1), \ldots, \varphi(e_d))$, so $\mathbf{b}$ is independent of $\mathbf{a}$, the above formula for φ becomes $\varphi(\mathbf{a}) = \mathbf{a} \cdot \mathbf{b}$. Thus each φ is the dot product on K^d with a specific vector. To prove, for each φ, that there is only one possible $\mathbf{b}$ for which $\varphi(\mathbf{a}) = \mathbf{a}\cdot\mathbf{b}$ for all $\mathbf{a}$, set $\mathbf{a} = e_i$ in this equation to get $\varphi(e_i) = e_i \cdot \mathbf{b}$. Since the dot product with e_i produces the ith coordinate, we see that the ith coordinate of $\mathbf{b}$ *must* be $\varphi(e_i)$, so as i varies we conclude there is only one possible choice of $\mathbf{b}$, namely the d-tuple we used to define $\mathbf{b}$ originally. $\square$

We want to extend Theorem 3.1 from the finite-dimensional spaces K^d to the infinite-dimensional space $c_0(K)$ when $(K, |\cdot|)$ is a complete valued field. This is possible if we impose a continuity condition.

Theorem 3.2. *Each continuous K-linear function $\varphi\colon c_0(K) \to K$ has the form $\varphi(\mathbf{a}) = \sum_{i\geq 1} a_i b_i$ for at most one sequence $(b_1, b_2, \ldots)$ in K.*

Proof. We saw in the proof of Corollary 2.3 that each $\mathbf{a} = (a_1, a_2, \ldots)$ in $c_0(K)$ is $\lim_{n\to\infty} \mathbf{a}_n$, where $\mathbf{a}_n = (a_1, \ldots, a_n, 0, 0, \ldots)$. Therefore if $\varphi\colon c_0(K) \to K$ is K-linear and continuous,

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a} \Longrightarrow \lim_{n\to\infty} \varphi(\mathbf{a}_n) = \varphi(\mathbf{a}). \tag{3.1}$$

Let $e_i = (0, \ldots, 0, 1, 0, \ldots)$ have 1 in its ith component and 0 in all other components. Then $e_i \in c_0(K)$ and

$$\mathbf{a}_n = (a_1, \ldots, a_n, 0, 0, \ldots) = a_1 e_1 + \cdots + a_n e_n,$$

which is a finite sum of terms, so by linearity

$$\varphi(\mathbf{a}_n) = \varphi(a_1 e_1 + \cdots + a_n e_n) = a_1\varphi(e_1) + \cdots + a_n\varphi(e_n).$$

Therefore by (3.1),

$$\varphi(\mathbf{a}) = \lim_{n\to\infty} \varphi(\mathbf{a}_n) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i\varphi(e_i).$$

By the definition of an infinite series as a limit of partial sums we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i\varphi(e_i) = \sum_{i\geq 1} a_i\varphi(e_i),$$

and this limit exists because it is a calculation of $\varphi(\mathbf{a})$. Setting $b_i = \varphi(e_i)$, which is independent of $\mathbf{a}$, we have for all $\mathbf{a}$ in $c_0(K)$ that

$$\varphi(\mathbf{a}) = \sum_{i\geq 1} a_i b_i. \tag{3.2}$$

Substituting $\mathbf{a} = e_j$ into (3.2), we recover $\varphi(e_j) = b_j$ for all j, so there is at most one sequence $\{b_i\}$ in K making the desired formula for φ in (3.2) work out. $\square$

In this proof we used continuity of φ to calculate $\varphi(\mathbf{a})$ from $\varphi(\mathbf{a}_n)$ by passage to a limit.

Something is missing from Theorem 3.2: a description of necessary constraints on a sequence $b_1, b_2, \ldots$ in K in order for the infinite-dimensional dot product

$$\mathbf{a}\cdot\mathbf{b} = \sum_{i\geq 1} a_i b_i \tag{3.3}$$

to converge for every $\mathbf{a} \in c_0(K)$. It is not true[2] that for every sequence $b_1, b_2, \ldots$ in K that the series in (3.3) converges for all $\mathbf{a} \in c_0(K)$. For example, if $x \in K$ satisfies $0 < |x| < 1$ then $\{x^n\} \in c_0(K)$ and the series in (3.3) doesn't converge when $\mathbf{a} = \{x^n\}$ and $\mathbf{b} = \{1/x^n\}$.

[2]Except when the absolute value on K is trivial, making $c_0(K)$ the sequences in K whose terms eventually equal 0.

When $K = \mathbf{R}$ or $K = \mathbf{Q}_p$, here are sufficient conditions on the b_i's for convergence of the dot products (3.3) as $\mathbf{a}$ runs over all of $c_0(K)$.

- In $\mathbf{R}$, if $\sum_{i\geq 1} |b_i|$ converges then $\sum_{i\geq 1} a_i b_i$ converges for all $\mathbf{a} \in c_0(\mathbf{R})$: $\{a_i\}$ is bounded, say $|a_i| \leq A$ for all i, so $|a_i b_i| \leq A|b_i|$ and therefore $\sum_{i\geq 1} |a_i b_i| \leq A\sum_{i\geq 1} |b_i| < \infty$, so $\sum_{i\geq 1} a_i b_i$ converges (absolutely) in $\mathbf{R}$.
- In $\mathbf{Q}_p$, if $\{b_i\}$ is bounded then $\sum_{i\geq 1} a_i b_i$ converges for all $\mathbf{a} \in c_0(\mathbf{Q}_p)$: letting $|a_i|_p \leq A$ we have $|a_i b_i|_p \leq A|b_i|_p$, and $A|b_i|_p \to 0$ as $i \to \infty$, so $a_i b_i \to 0$ in $\mathbf{Q}_p$ as $i \to \infty$. Thus $\sum_{n\geq 1} a_i b_i$ converges in $\mathbf{Q}_p$.

Perhaps surprisingly, these sufficient conditions turn out to be necessary as well. That's what we show in the next two theorems.

Theorem 3.3. *Let $\{b_i\}$ be a sequence in $\mathbf{R}$ such that for all $\mathbf{a} \in c_0(\mathbf{R})$, $\sum_{i\geq 1} a_i b_i$ converges. Then $\sum_{i\geq 1} |b_i|$ converges. Moreover, the function $c_0(\mathbf{R}) \to \mathbf{R}$ defined by $\{a_i\} \mapsto \sum_{i\geq 1} a_i b_i$ is continuous.*

Proof. We will prove the contrapositive: if $\sum_{i\geq 1} |b_i|$ does not converge then there is an $\mathbf{a} \in c_0(\mathbf{R})$ such that $\sum_{i\geq 1} a_i b_i$ does not converge. The construction we use is one I learned from Iddo Ben-Ari.

Since $\sum_{i\geq 1} |b_i|$ is a series of nonnegative numbers, that it does not converge means it is ∞. Therefore letting $s_i = |b_1| + |b_2| + \cdots + |b_i|$, we have $s_i \leq s_{i+1}$ for all i and $s_i \to \infty$. Choose j minimal so that $s_j > 0$ (maybe some initial b_i's are 0). For $i \geq j$ define $a_i = \mathrm{sign}(b_i)/\sqrt{s_i}$, and for $i < j$ define $a_i = 0$. Then $a_i \to 0$ since $s_i \to \infty$, and for $N \geq j$ we have

$$\begin{aligned}
\sum_{i=1}^{N} a_i b_i &= \sum_{i=j}^{N} a_i b_i \quad \text{since } b_i = 0 \text{ for } i < j \\
&= \sum_{i=j}^{N} \frac{\mathrm{sign}(b_i) b_i}{\sqrt{s_i}} \\
&= \sum_{i=j}^{N} \frac{|b_i|}{\sqrt{s_i}} \\
&\geq \sum_{i=j}^{N} \frac{|b_i|}{\sqrt{s_N}} \quad \text{since } s_i \leq s_N \text{ when } i \leq N \text{ and } |b_i| \geq 0 \\
&= \frac{1}{\sqrt{s_N}} \sum_{i=1}^{N} |b_i| \quad \text{since } |b_i| = 0 \text{ for } i < j \\
&= \frac{s_N}{\sqrt{s_N}} \\
&= \sqrt{s_N},
\end{aligned}$$

which tends to ∞ as $N \to \infty$, so $\sum_{i\geq 1} a_i b_i$ does not converge.

When $\sum_{i\geq 1} |b_i|$ converges, let S be the value of this series. Then for $\mathbf{a}$ and $\mathbf{a}'$ in $c_0(\mathbf{R})$,

$$\left|\mathbf{a} \cdot \mathbf{b} - \mathbf{a}' \cdot \mathbf{b}\right| = \left|\sum_{i\geq 1} (a_i - a_i')b_i\right| \leq \sum_{i\geq 1} |a_i - a_i'||b_i| \leq ||\mathbf{a} - \mathbf{a}'|| \sum_{i\geq 1} |b_i| \leq S||\mathbf{a} - \mathbf{a}'||.$$

That shows $\mathbf{a} \mapsto \mathbf{a} \cdot \mathbf{b}$ is a (uniformly) continuous function $c_0(\mathbf{R}) \to \mathbf{R}$. □

Theorem 3.4. *Let $\{b_i\}$ be a sequence in $\mathbf{Q}_p$ such that for all $\mathbf{a} \in c_0(\mathbf{Q}_p)$, $\sum_{i\geq 1} a_i b_i$ converges. Then $\{b_i\}$ is bounded. Moreover, the function $c_0(\mathbf{Q}_p) \to \mathbf{Q}_p$ defined by $\{a_i\} \mapsto \sum_{i\geq 1} a_i b_i$ is continuous.*

Proof. As in the previous proof we will prove the contrapositive: if $\{b_i\}$ is unbounded in $\mathbf{Q}_p$ then there is an $\mathbf{a} \in c_0(\mathbf{Q}_p)$ such that $\sum_{i\geq 1} a_i b_i$ does not converge.

That $\{b_i\}$ is unbounded does not mean $|b_i|_p \to \infty$, but only that there is a *subsequence* of $\{b_i\}$ whose terms have absolute value tending to ∞. Thus there is a subsequence $\{b_{i_k}\}$ such that $|b_{i_k}|_p \geq k$ for each $k \geq 1$. Define a_i by $a_{i_k} = 1/b_{i_k}$ if i is some i_k and $a_i = 0$ if i is not an i_k. Then $|a_{i_k}|_p = 1/|b_{i_k}|_p \leq 1/k$, while $|a_i| = 0$ if i is not an i_k, so $a_i \to 0$ as $i \to \infty$ and

$$\sum_{i\geq 1} a_i b_i = \sum_{k\geq 1} a_{i_k} b_{i_k} = \sum_{k\geq 1} 1,$$

which does not converge.

When $\{b_i\}$ is bounded in $\mathbf{Q}_p$, let $M = \max_{i\geq 1} |b_i|_p$. For $\mathbf{a}$ and $\mathbf{a}'$ in $c_0(\mathbf{Q}_p)$,

$$\left|\mathbf{a} \cdot \mathbf{b} - \mathbf{a}' \cdot \mathbf{b}\right|_p = \left|\sum_{i\geq 1} (a_i - a_i')b_i\right|_p \leq \max_{i\geq 1} |a_i - a_i'|_p |b_i|_p \leq M||\mathbf{a} - \mathbf{a}'||,$$

so $\mathbf{a} \mapsto \mathbf{a} \cdot \mathbf{b}$ is a (uniformly) continuous function $c_0(\mathbf{Q}_p) \to \mathbf{Q}_p$. □

Let's summarize what we have shown:

(1) For every complete valued field $(K, |\cdot|)$, the continuous K-linear maps $c_0(K) \to K$ can all be described as taking a dot product on $c_0(K)$ with a unique sequence in K.
(2) The continuous $\mathbf{R}$-linear maps $c_0(\mathbf{R}) \to \mathbf{R}$ are the dot products on $c_0(\mathbf{R})$ with a sequence $\{b_i\}$ in $\mathbf{R}$ such that $\sum_{i\geq 1} |b_i|$ converges.
(3) The continuous $\mathbf{Q}_p$-linear maps $c_0(\mathbf{Q}_p) \to \mathbf{Q}_p$ are the dot products on $c_0(\mathbf{Q}_p)$ with a bounded sequence in $\mathbf{Q}_p$.

It is left to the reader to check that the second point remains true when $\mathbf{R}$ replaced by $\mathbf{C}$ and the third point remains true when $\mathbf{Q}_p$ is replaced by a nonarchimedean complete valued field. Since, by a theorem of Ostrowski, every complete valued field is either $\mathbf{R}$, $\mathbf{C}$, or its absolute value is non-archimedean, we have in fact described the continuous K-linear maps $c_0(K) \to K$ for all possible complete valued fields $(K, |\cdot|)$.[3]

Remark 3.5. The proofs that $c_0(K)$ and the space $c_b(K)$ of bounded sequences in K are both complete metric spaces (using the same formula for the metric) are quite similar, but the proof of Theorem 3.2 does not carry over to $c_b(K)$ and, in fact, the space of all continuous linear mappings $c_b(\mathbf{R}) \to \mathbf{R}$ does not admit an elementary description in terms of dot products.

[3]When K is given the trivial absolute value, $c_0(K)$ is the sequences in K that are eventually 0 and all sequences in K are bounded.