

SERIES FOR THE LOGARITHM AND ARCTANGENT

KEITH CONRAD

1. INTRODUCTION

The purpose of this handout is twofold: review the derivation of the identities

$$(1.1) \quad \log(1+x) = \sum_{n \geq 1} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$$

and

$$(1.2) \quad \arctan x = \sum_{n \geq 0} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \dots$$

for $0 \leq x \leq 1$ and extend them to a range which includes negative values of x . Specifically, we want to show (1.1) is true for $-1 < x \leq 1$ and (1.2) is true for $-1 \leq x \leq 1$.

The main idea in both cases is an integration formula:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}, \quad \arctan x = \int_0^x \frac{dt}{1+t^2}.$$

These formulas connect both functions with the simpler function (integrand) $1/(1-t)$ under a change of variables.

2. GEOMETRIC SERIES AND INTEGRATION

We start with the finite geometric series

$$1 + t + t^2 + \dots + t^N = \frac{t^{N+1} - 1}{t - 1} = \frac{1 - t^{N+1}}{1 - t}.$$

Here $t \neq 1$. We isolate the term $1/(1-t)$:

$$(2.1) \quad \frac{1}{1-t} = 1 + t + t^2 + \dots + t^N + \frac{t^{N+1}}{1-t}.$$

Again, this is valid for $t \neq 1$. Now we make two different changes of variable to turn the left side of (2.1) into an integrand adapted to the logarithm and arctangent functions. Replacing t with $-t$ in (2.1) gives

$$(2.2) \quad \begin{aligned} \frac{1}{1+t} &= 1 - t + t^2 + \dots + (-1)^N t^N + (-1)^{N+1} \frac{t^{N+1}}{1+t} \\ &= \sum_{n=0}^N (-1)^n t^n + (-1)^{N+1} \frac{t^{N+1}}{1+t}. \end{aligned}$$

Here we must have $t \neq -1$.

Replacing t with $-t^2$ in (2.1) gives

$$(2.3) \quad \begin{aligned} \frac{1}{1+t^2} &= 1 - t^2 + t^4 - \cdots + (-1)^N t^{2N} + (-1)^{N+1} \frac{t^{2(N+1)}}{1+t^2} \\ &= \sum_{n=0}^N (-1)^n t^{2n} + (-1)^{N+1} \frac{t^{2(N+1)}}{1+t^2}. \end{aligned}$$

Here t can be any number: the denominator doesn't become 0 since t^2 is never -1 .

These formulas for $1/(1+t)$ and for $1/(1+t^2)$ will now be integrated from 0 to x (for variable x). Well, we have to make sure -1 is not between 0 and x in the case of (2.2) since $1/(1+t)$ blows up at $t = -1$. We will apply $\int_0^x (\cdots) dt$ to (2.2) with $x > -1$. We obtain

$$\int_0^x \frac{dt}{1+t} = \sum_{n=0}^N (-1)^n \int_0^x t^n dt + (-1)^{N+1} \int_0^x \frac{t^{N+1}}{1+t} dt,$$

which is the same as

$$(2.4) \quad \begin{aligned} \log(1+x) &= \sum_{n=0}^N (-1)^n \frac{x^{n+1}}{n+1} + (-1)^{N+1} \int_0^x \frac{t^{N+1}}{1+t} dt \\ &= \sum_{n=1}^{N+1} (-1)^{n-1} \frac{x^n}{n} + (-1)^{N+1} \int_0^x \frac{t^{N+1}}{1+t} dt. \end{aligned}$$

Now put the sum on one side and the other terms on the other side:

$$(2.5) \quad \sum_{n=1}^{N+1} (-1)^{n-1} \frac{x^n}{n} = \log(1+x) + (-1)^N \int_0^x \frac{t^{N+1}}{1+t} dt$$

Similarly, applying $\int_0^x (\cdots) dt$ to (2.3) gives

$$\int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^N (-1)^n \int_0^x t^{2n} dt + (-1)^{N+1} \int_0^x \frac{t^{2(N+1)}}{1+t^2} dt,$$

which is the same as

$$\arctan x = \sum_{n=0}^N (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{N+1} \int_0^x \frac{t^{2(N+1)}}{1+t^2} dt.$$

Isolating the sum on one side yields, after some algebra,

$$(2.6) \quad \sum_{n=0}^N (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + (-1)^N \int_0^x \frac{t^{2(N+1)}}{1+t^2} dt.$$

Our remaining task is to show, for *suitable* fixed x , that the definite integrals on the right sides of (2.5) and (2.6) tend to 0 as $N \rightarrow \infty$. When that happens we obtain the series representations (1.1) and (1.2) for those values of x .

From this point of view, the important issue concerning the definite integrals in (2.5) and (2.6) is not to compute them exactly, but to *estimate* them and show in favorable cases through these estimates that the integrals tend to 0 as $N \rightarrow \infty$.

3. THE LOGARITHM CASE

For the logarithm series (2.5), we will show

$$(3.1) \quad \lim_{N \rightarrow \infty} \int_0^x \frac{t^{N+1}}{1+t} dt = 0$$

for each x in the half-open interval $(-1, 1]$.

First suppose $0 \leq x \leq 1$. When $t \in [0, x]$ we have

$$0 \leq \frac{t^{N+1}}{1+t} \leq t^{N+1}.$$

Integrating these inequalities over the interval $[0, x]$ gives

$$0 \leq \int_0^x \frac{t^{N+1}}{1+t} dt \leq \int_0^x t^{N+1} dt = \frac{x^{N+2}}{N+2} \leq \frac{1}{N+2}.$$

Our integral of interest is squeezed between 0 and $1/(N+2)$. As $N \rightarrow \infty$ the upper bound tends to 0, so (3.1) has been verified when $0 \leq x \leq 1$.

Now suppose $-1 < x < 0$. Write $x = -y$ with $0 < y < 1$. Then, using the change of variables $u = -t$,

$$\begin{aligned} \int_0^x \frac{t^{N+1}}{1+t} dt &= (-1)^N \int_0^{-x} \frac{u^{N+1}}{1-u} du \\ &= (-1)^N \int_0^y \frac{u^{N+1}}{1-u} du. \end{aligned}$$

The integrand $u^{N+1}/(1-u)$ is nonnegative for $0 \leq u \leq y$, so

$$\left| \int_0^x \frac{t^{N+1}}{1+t} dt \right| = \int_0^y \frac{u^{N+1}}{1-u} du.$$

To get an upper bound on the integral we replace the denominator $1-u$ by the lower bound $1-y$:

$$0 \leq u \leq y \implies \frac{u^{N+1}}{1-u} \leq \frac{u^{N+1}}{1-y}.$$

Therefore

$$(3.2) \quad \left| \int_0^x \frac{t^{N+1}}{1+t} dt \right| = \int_0^y \frac{u^{N+1}}{1-u} du \leq \int_0^y \frac{u^{N+1}}{1-y} du.$$

Since $0 < y < 1$,

$$\begin{aligned} \int_0^y \frac{u^{N+1}}{1-y} du &= \frac{1}{1-y} \int_0^y u^{N+1} du \\ &= \frac{y^{N+2}}{(N+2)(1-y)} \\ &\leq \frac{1}{(N+2)(1-y)} \\ &= \frac{1}{(N+2)(1+x)}. \end{aligned}$$

Feeding this back into (3.2) gives

$$\left| \int_0^x \frac{t^{N+1}}{1+t} dt \right| \leq \frac{1}{(N+2)(1+x)}.$$

For each x in $(-1, 0)$, this upper bound tends to 0 as $N \rightarrow \infty$.

We have completed the derivation of (1.1) for $-1 < x \leq 1$.

4. THE ARCTANGENT CASE

In (2.6) we want to show

$$\lim_{N \rightarrow \infty} \int_0^x \frac{t^{2(N+1)}}{1+t^2} dt = 0$$

for each x in the closed interval $[-1, 1]$.

First we suppose $0 \leq x \leq 1$. If $0 \leq t \leq x$ then

$$0 \leq \frac{t^{2(N+1)}}{1+t^2} \leq t^{2(N+1)},$$

so

$$0 \leq \int_0^x \frac{t^{2(N+1)}}{1+t^2} dt \leq \int_0^x t^{2(N+1)} dt = \frac{x^{2N+3}}{2N+3} \leq \frac{1}{2N+3}.$$

As $N \rightarrow \infty$, the upper bound tends to 0 so the integral tends to 0.

Now suppose $-1 \leq x < 0$. Write $x = -y$ where $0 < y \leq 1$. Then, using the substitution $u = -t$,

$$\int_0^x \frac{t^{2(N+1)}}{1+t^2} dt = - \int_0^{-x} \frac{u^{2(N+1)}}{1+u^2} du = - \int_0^y \frac{u^{2(N+1)}}{1+u^2} du.$$

The integral on the right is exactly the kind we treated already: as $N \rightarrow \infty$ it tends to 0.

This completes the derivation of (1.2) for $-1 \leq x \leq 1$.