SERIES FOR THE LOGARITHM AND ARCTANGENT

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1. Introduction

The purpose of this handout is twofold: review the derivation of the identities

$$\log(1+x) = \sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots \tag{1.1}$$

and

$$\arctan x = \sum_{n\geq 0}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots \tag{1.2}$$

for $0 \leq x \leq 1$ and extend them to a range which includes negative values of x. Specifically, we want to show (1.1) is true for $-1 < x \leq 1$ and (1.2) is true for $-1 \leq x \leq 1$.

The main idea in both cases is an integration formula:

$$\log(1+x) = \int_0^x \frac{\mathrm{d}t}{1+t}, \quad \arctan x = \int_0^x \frac{\mathrm{d}t}{1+t^2}.$$

These formulas connect both functions with the simpler function (integrand) $1/(1-t)$ under a change of variables.

2. Geometric series and integration

We start with the finite geometric series

$$1 + t + t^2 + \cdots + t^N = \frac{t^{N+1}-1}{t-1} = \frac{1-t^{N+1}}{1-t}.$$

Here $t \neq 1$. We isolate the term $1/(1-t)$:

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^N + \frac{t^{N+1}}{1-t}. \tag{2.1}$$

Again, this is valid for $t \neq 1$. Now we make two different changes of variable to turn the left side of (2.1) into an integrand adapted to the logarithm and arctangent functions. Replacing t with $-t$ in (2.1) gives

$$\begin{aligned}\frac{1}{1+t} &= 1 - t + t^2 + \cdots + (-1)^N t^N + (-1)^{N+1}\frac{t^{N+1}}{1+t} \\ &= \sum_{n=0}^{N}(-1)^n t^n + (-1)^{N+1}\frac{t^{N+1}}{1+t}. \qquad (2.2)\end{aligned}$$

Here we must have $t \neq -1$.

Replacing t with $-t^2$ in (2.1) gives

$$\begin{aligned} \frac{1}{1+t^2} &= 1 - t^2 + t^4 + \cdots + (-1)^N t^{2N} + (-1)^{N+1}\frac{t^{2(N+1)}}{1+t^2} \\ &= \sum_{n=0}^{N} (-1)^n t^{2n} + (-1)^{N+1}\frac{t^{2(N+1)}}{1+t^2}. \end{aligned} \tag{2.3}$$

Here t can be any number: the denominator doesn't become 0 since t^2 is never -1.

These formulas for $1/(1+t)$ and for $1/(1+t^2)$ will now be integrated from 0 to x (for variable x). Well, we have to make sure -1 is not between 0 and x in the case of (2.2) since $1/(1+t)$ blows up at $t = -1$. We will apply $\int_0^x (\cdots)\,\mathrm{d}t$ to (2.2) with $x > -1$. We obtain

$$\int_0^x \frac{\mathrm{d}t}{1+t} = \sum_{n=0}^{N} (-1)^n \int_0^x t^n\,\mathrm{d}t + (-1)^{N+1}\int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t,$$

which is the same as

$$\begin{aligned} \log(1+x) &= \sum_{n=0}^{N} (-1)^n \frac{x^{n+1}}{n+1} + (-1)^{N+1}\int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t \\ &= \sum_{n=1}^{N+1} (-1)^{n-1} \frac{x^n}{n} + (-1)^{N+1}\int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t. \end{aligned} \tag{2.4}$$

Now put the sum on one side and the other terms on the other side:

$$\sum_{n=1}^{N+1} (-1)^{n-1} \frac{x^n}{n} = \log(1+x) + (-1)^N \int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t \tag{2.5}$$

Similarly, applying $\int_0^x (\cdots)\,\mathrm{d}t$ to (2.3) gives

$$\int_0^x \frac{\mathrm{d}t}{1+t^2} = \sum_{n=0}^{N} (-1)^n \int_0^x t^{2n}\,\mathrm{d}t + (-1)^{N+1}\int_0^x \frac{t^{2(N+1)}}{1+t^2}\,\mathrm{d}t,$$

which is the same as

$$\arctan x = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{N+1}\int_0^x \frac{t^{2(N+1)}}{1+t^2}\,\mathrm{d}t.$$

Isolating the sum on one side yields, after some algebra,

$$\sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + (-1)^N \int_0^x \frac{t^{2(N+1)}}{1+t^2}\,\mathrm{d}t. \tag{2.6}$$

Our remaining task is to show, for *suitable* fixed x, that the definite integrals on the right sides of (2.5) and (2.6) tend to 0 as $N \to \infty$. When that happens we obtain the series representations (1.1) and (1.2) for those values of x.

From this point of view, the important issue concerning the definite integrals in (2.5) and (2.6) is not to compute them exactly, but to *estimate* them and show in favorable cases through these estimates that the integrals tend to 0 as $N \to \infty$.

3. The logarithm case

For the logarithm series (2.5), we will show

$$\lim_{N\to\infty}\int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t = 0 \tag{3.1}$$

for each x in the half-open interval $(-1,1]$.

First suppose $0 \le x \le 1$. When $t \in [0,x]$ we have

$$0 \le \frac{t^{N+1}}{1+t} \le t^{N+1}.$$

Integrating these inequalities over the interval $[0,x]$ gives

$$0 \le \int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t \le \int_0^x t^{N+1}\,\mathrm{d}t = \frac{x^{N+2}}{N+2} \le \frac{1}{N+2}.$$

Our integral of interest is squeezed between 0 and $1/(N+2)$. As $N\to\infty$ the upper bound tends to 0, so (3.1) has been verified when $0 \le x \le 1$.

Now suppose $-1 < x < 0$. Write $x = -y$ with $0 < y < 1$. Then, using the change of variables $u = -t$,

$$\begin{aligned}\int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t &= (-1)^N \int_0^{-x} \frac{u^{N+1}}{1-u}\,\mathrm{d}u \\ &= (-1)^N \int_0^{y} \frac{u^{N+1}}{1-u}\,\mathrm{d}u.\end{aligned}$$

The integrand $u^{N+1}/(1-u)$ is nonnegative for $0 \le u \le y$, so

$$\left|\int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t\right| = \int_0^y \frac{u^{N+1}}{1-u}\,\mathrm{d}u.$$

To get an upper bound on the integral we replace the denominator $1-u$ by the lower bound $1-y$:

$$0 \le u \le y \Longrightarrow \frac{u^{N+1}}{1-u} \le \frac{u^{N+1}}{1-y}.$$

Therefore

$$\left|\int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t\right| = \int_0^y \frac{u^{N+1}}{1-u}\,\mathrm{d}u \le \int_0^y \frac{u^{N+1}}{1-y}\,\mathrm{d}u. \tag{3.2}$$

Since $0 < y < 1$,

$$\begin{aligned}\int_0^y \frac{u^{N+1}}{1-y}\,\mathrm{d}u &= \frac{1}{1-y}\int_0^y u^{N+1}\,\mathrm{d}u \\ &= \frac{y^{N+2}}{(N+2)(1-y)} \\ &\le \frac{1}{(N+2)(1-y)} \\ &= \frac{1}{(N+2)(1+x)}.\end{aligned}$$

Feeding this back into (3.2) gives

$$\left|\int_0^x \frac{t^{N+1}}{1+t}\,\mathrm{d}t\right| \leq \frac{1}{(N+2)(1+x)}.$$

For each x in $(-1,0)$, this upper bound tends to 0 as $N \to \infty$.

We have completed the derivation of (1.1) for $-1 < x \leq 1$.

4. The arctangent case

In (2.6) we want to show

$$\lim_{N\to\infty} \int_0^x \frac{t^{2(N+1)}}{1+t^2}\,\mathrm{d}t = 0$$

for each x in the closed interval $[-1,1]$.

First we suppose $0 \leq x \leq 1$. If $0 \leq t \leq x$ then

$$0 \leq \frac{t^{2(N+1)}}{1+t^2} \leq t^{2(N+1)},$$

so

$$0 \leq \int_0^x \frac{t^{2(N+1)}}{1+t^2}\,\mathrm{d}t \leq \int_0^x t^{2(N+1)}\,\mathrm{d}t = \frac{x^{2N+3}}{2N+3} \leq \frac{1}{2N+3}.$$

As $N \to \infty$, the upper bound tends to 0 so the integral tends to 0.

Now suppose $-1 \leq x < 0$. Write $x = -y$ where $0 < y \leq 1$. Then, using the substitution $u = -t$,

$$\int_0^x \frac{t^{2(N+1)}}{1+t^2}\,\mathrm{d}t = -\int_0^{-x} \frac{u^{2(N+1)}}{1+u^2}\,\mathrm{d}u = -\int_0^y \frac{u^{2(N+1)}}{1+u^2}\,\mathrm{d}u.$$

The integral on the right is exactly the kind we treated already: as $N \to \infty$ it tends to 0.

This completes the derivation of (1.2) for $-1 \leq x \leq 1$.