IRRATIONALITY OF π AND e

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1. Introduction

Numerical estimates for π have been found in records of several ancient civilizations. These estimates were all based on inscribing and circumscribing regular polygons around a circle to get upper and lower bounds on the area (and thus upper and lower bounds on π after dividing the area by the square of the radius). Such estimates are accurate to a few decimal places. Around 1600, Ludolph van Ceulen gave an estimate for π to 35 decimal places. He spent many years of his life on this calculation, using a polygon with 2^{62} sides!

With the advent of calculus in the 17-th century, a new approach to the calculation of π became available: infinite series. For instance, if we integrate

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \cdots, \qquad |t| < 1$$

from $t = 0$ to $t = x$ when $|x| < 1$, we find

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots. \tag{1.1}$$

Actually, this is also correct at the boundary point $x = 1$. Since $\arctan 1 = \pi/4$, (1.1) specializes to the formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots, \tag{1.2}$$

which is due to Leibniz. It expresses π in terms of an alternating sum of reciprocals of odd numbers. However, the series in (1.2) converges too slowly to be numerically useful. For example, truncating the series after 1000 terms and multiplying by 4 gives the approximation $\pi \approx 3.1405$, which is only good to two places after the decimal point.

There are other formulas for π in terms of arctan values, such as

$$\begin{aligned}
\frac{\pi}{4} &= \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) \\
&= 2\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) \\
&= 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).
\end{aligned}$$

Since the series for $\arctan x$ is more rapidly convergent when x is less than 1, these other series are more useful than (1.2) to get good numerical approximations to π. The last such calculation before the use of computers was by Shanks in 1873. He claimed to have found π to 707 places. In the 1940s, the first computer estimate for π revealed that Shanks made a mistake in the 528-th digit, so all his further calculations were in error!

Our interest here is not to ponder ever more elaborate methods of estimating π, but to prove something about the *structure* of this number: it is irrational. That is, π is not a ratio of integers. The idea of the proof is to argue by contradiction. This is also the principle behind the simpler proof that the number $\sqrt{2}$ is irrational. However, there is an essential difference between proofs that $\sqrt{2}$ is irrational and proofs that π is irrational. One can prove $\sqrt{2}$ is irrational using only algebraic manipulations with a hypothetical rational expression for $\sqrt{2}$ to reach a contradiction. But all known proofs of the irrationality of π are based on techniques from calculus, which can be used to prove irrationality of other numbers, such as e and rational powers of e (aside from $e^0 = 1$).

The remaining sections are organized as follows. In Section 2, we prove π is irrational with definite integrals. Irrationality of e is proved by infinite series in Section 3. A general discussion about irrationality proofs is in Section 4, and we apply those ideas to prove irrationality of nonzero rational powers of e in Section 5. In Section 6 we introduce complex numbers into a proof from Section 5 in order to obtain another proof that π is irrational.

2. Irrationality of π

The first proof that π is irrational is due to Lambert in 1761. His proof involves an analytic device that is not covered in calculus courses: continued fractions. (A discussion of this work is in [3, pp. 68–78].) The irrationality proof for π we give here is due to Niven [5] and uses integrals instead of continued fractions.

Theorem 2.1. *The number π is irrational.*

Proof. For a nice function $f(x)$, a double integration by parts shows

$$\int f(x)\sin x \,\mathrm{d}x = -f(x)\cos x + f'(x)\sin x - \int f''(x)\sin x\,\mathrm{d}x.$$

Therefore (using $\sin(0) = 0$, $\cos(0) = 1$, $\sin(\pi) = 0$, and $\cos(\pi) = -1$),

$$\int_0^\pi f(x)\sin x\,\mathrm{d}x = (f(0) + f(\pi)) - \int_0^\pi f''(x)\sin x\,\mathrm{d}x.$$

In particular, if $f(x)$ is a polynomial of even degree, say $2n$, then repeating this calculation n times gives

$$\int_0^\pi f(x)\sin x\,\mathrm{d}x = F(0) + F(\pi), \tag{2.1}$$

where $F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$.

To prove π is irrational, we argue by contradiction. Assume $\pi = p/q$ for positive integers p and q (since $\pi > 0$). We are going to apply (2.1) to a carefully (and mysteriously!) chosen polynomial $f(x)$ and wind up with an integer that lies between 0 and 1. No such integer exists, so we have a contradiction and therefore π is irrational.

For a positive integer n, set

$$f_n(x) = q^n\frac{x^n(\pi - x)^n}{n!} = \frac{x^n(p - qx)^n}{n!}. \tag{2.2}$$

This polynomial of degree $2n$ depends on n (and on π!). We are going to apply (2.1) to this polynomial and find a contradiction when n becomes large.

Before working out the consequences of (2.1) for $f(x) = f_n(x)$, we note the polynomial $f_n(x)$ has two important properties:

- for $0 < x < \pi$, $f_n(x)$ is positive and (when n is large) very small,
- all the derivatives of $f_n(x)$ at $x = 0$ and $x = \pi$ are integers.

To show the first property is true, the positivity of $f_n(x)$ for $0 < x < \pi$ is immediate from its defining formula. To bound $f_n(x)$ from above when $0 < x < \pi$, note that $0 < \pi - x < \pi$, so $0 < x(\pi - x) < \pi^2$. Therefore

$$0 < f_n(x) \leq q^n \left(\frac{\pi^{2n}}{n!}\right) = \frac{(q\pi^2)^n}{n!}. \tag{2.3}$$

The upper bound tends to 0 as $n \to \infty$, so the upper bound is less than 1 for large n.

To show the second property is true, first take $x = 0$. The coefficient of x^j in $f_n(x)$ is $f_n^{(j)}(0)/j!$. Since $f_n(x) = x^n(p - qx)^n/n!$ and p and q are integers, the binomial theorem tells us the coefficient of x^j can also be written as $c_j/n!$ for an integer c_j. Therefore

$$f_n^{(j)}(0) = \frac{j!}{n!}c_j. \tag{2.4}$$

Since $f_n(x)$ has its lowest degree nonvanishing term in degree n, $c_j = 0$ for $j < n$, so $f_n^{(j)}(0) = 0$ for $j < n$. For $j \geq n$, $j!/n!$ is an integer, so $f_n^{(j)}(0)$ is an integer by (2.4).

To see the derivatives of $f_n(x)$ at $x = \pi$ are also integers, we use the identity $f_n(\pi - x) = f_n(x)$. Differentiate both sides j times and set $x = 0$ to get $(-1)^j f_n^{(j)}(\pi) = f_n^{(j)}(0)$ for all j. Therefore, since the right side is an integer, the left side is an integer too. This concludes the proof of the two important properties of $f_n(x)$.

Now we look at (2.1) when $f = f_n$. Since all derivatives of f_n at 0 and π are integers, the right side of (2.1) is an integer when $f = f_n$ (look at the definition of $F(x)$). Therefore $\int_0^\pi f_n(x)\sin x\,\mathrm{d}x$ is an integer for every n. Since $f_n(x)$ and $\sin x$ are positive on $(0, \pi)$, this integral is a positive integer. However, when n is large, $|f_n(x)\sin x| \leq |f_n(x)| \leq (q\pi^2)^n/n!$ by (2.3). As $n \to \infty$, $(q\pi^2)^n/n! \to 0$. Therefore $\int_0^\pi f_n(x)\sin x\,\mathrm{d}x$ is a positive integer less than 1 when n is very large. This is absurd, so we have reached a contradiction. Thus π is irrational. □

This proof is quite puzzling. How did Niven choose the polynomials $f_n(x)$ or know to compute the integral (2.1)? Here is a reworking of Niven's proof in terms of recursions, due to Markov and Zhou [8].

Proof. Set

$$I_n = \int_0^\pi \frac{(x(\pi - x))^n}{n!}\sin x\,\mathrm{d}x.$$

The integrand is continuous and positive on $(0, \pi)$, so $I_n > 0$. By explicit calculation, $I_0 = 2$ and $I_1 = 4$:

$$I_0 = \int_0^\pi \sin x\,\mathrm{d}x = -\cos x\Big|_0^\pi = -(-1) + 1 = 2$$

and

$$\begin{aligned}
I_1 &= \int_0^\pi (\pi x - x^2)\sin x\,\mathrm{d}x \\
&= \pi\int_0^\pi x\sin x\,\mathrm{d}x - \int_0^\pi x^2\sin x\,\mathrm{d}x \\
&= \pi(\sin x - x\cos x)\Big|_0^\pi - (2x\sin x - (x^2 - 2)\cos x)\Big|_0^\pi \\
&= \pi(\pi - 0) - ((\pi^2 - 2) - 2) \\
&= 4.
\end{aligned}$$

Using integration by parts twice, for $n \geq 2$ we have $I_n = (4n-2)I_{n-1} - \pi^2 I_{n-2}$, so by induction I_n is a polynomial in π^n of degree at most n with *integral* coefficients:

$$I_n = c_{n,n}\pi^n + c_{n-1,n}\pi^{n-1} + \cdots + c_{1,n}\pi + c_{0,n}. \tag{2.5}$$

The table below gives the first few explicit values of I_n to illustrate this formula, although we don't need to know them except for I_0 and I_1.

n	I_n
0	2
1	4
2	$-2\pi^2 + 24$
3	$-24\pi^2 + 240$
4	$2\pi^4 - 360\pi^2 + 3360$
5	$60\pi^4 - 6720\pi^2 + 60480$
6	$-2\pi^6 + 1680\pi^4 - 151200\pi^2 + 1330560$

Assume π is rational, so we can write $\pi = p/q$ for positive integers p and q. By (2.5), $q^n I_n = \sum_{k=0}^n c_{k,n} q^n \pi^k = \sum_{k=0}^n c_{k,n} q^{n-k} p^k \in \mathbf{Z}$. Since $I_n > 0$, we have $q^n I_n \in \mathbf{Z}^+$. At the same time, since $|x(\pi - x)| \leq \pi^2$ for $0 \leq x \leq \pi$ we have the bound $|q^n I_n| \leq q^n \int_0^\pi ((\pi^2)^n/n!)\,\mathrm{d}x = \pi((q\pi^2)^n/n!)$, so $I_n \to 0$ as $n \to \infty$. This contradicts $q^n I_n$ being a positive integer, so we have a contradiction. $\square$

3. Irrationality of e

We turn now to a proof that e is irrational. This was first established by Euler in 1737 using continued fractions. We will prove the irrationality in a more direct manner, using infinite series, by an argument of Fourier (1815).

Theorem 3.1. *The number e is irrational.*

Proof. Write

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

For a positive integer n,

$$\begin{aligned} e &= \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) + \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right) \\ &= \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) + \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+2)(n+1)} + \cdots\right). \end{aligned}$$

The second term in parentheses is positive and bounded above by the geometric series

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots = \frac{1}{n}.$$

Therefore

$$0 < e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) \leq \frac{1}{n \cdot n!}.$$

Write the sum $1 + 1 + 1/2! + \cdots + 1/n!$ as a fraction with common denominator $n!$: it is $p_n/n!$ with $p_n \in \mathbf{Z}$. Clear the denominator $n!$ to get

$$0 < n!e - p_n \leq \frac{1}{n}. \tag{3.1}$$

So far everything we have done involves no unproved assumptions. Now we introduce the rationality assumption. If e is rational, then $n!e$ is an integer when n is large (since each positive integer is a factor of $n!$ for all large n). But that makes $n!e-p_n$ an integer located in the open interval $(0, 1/n)$, which is absurd. We have a contradiction, so e is irrational. $\square$

4. General Ideas

It's time to think more systematically. A basic principle we need to understand is that numbers can be proved to be irrational if they can be approximated "too well" by rationals. Of course each number can be approximated arbitrarily closely by rational numbers: use a truncated decimal expansion. For instance, we can approximate $\sqrt{2} = 1.41421356...$ by

$$\frac{14142}{10000} = 1.4142, \quad \frac{1414213}{1000000} = 1.414213. \tag{4.1}$$

With truncated decimals, we achieve close estimates at the expense of rather large denominators. To see what this is all about, compare the above approximations with

$$\frac{99}{70} = 1.41428571..., \quad \frac{1393}{985} = 1.41421319..., \tag{4.2}$$

where we have achieved just as close an approximation with much smaller denominators (*e.g.*, the second one is accurate to 6 decimal places with a denominator of only 3 digits). These rational approximations to $\sqrt{2}$ are, in the sense of denominators, much better than the ones we find from decimal truncation.

To measure the "quality" of an approximation of a real number α by a rational number p/q, we should think not about the difference $|\alpha - p/q|$ being small in an absolute sense, but about the difference being substantially smaller than $1/q$ (thus tying the error with the size of the denominator in the approximation). In other words, we want

$$\frac{|\alpha - p/q|}{1/q} = q\left|\alpha - \frac{p}{q}\right| = |q\alpha - p|$$

to be small in an absolute sense.

Measuring the approximation of α by p/q using $|q\alpha - p|$ rather than $|\alpha - p/q|$ admittedly takes some getting used to, if you are new to the idea. Consider what it says about our approximations to $\sqrt{2}$. For example, from (4.1) we have

$$|10000\sqrt{2} - 14142| = .135623, \quad |1000000\sqrt{2} - 1414213| = .562373,$$

and these are not small when measured against $1/10000 = .0001$ or $1/1000000 = .000001$. On the other hand, from the approximations to $\sqrt{2}$ in (4.2) we have

$$|70\sqrt{2} - 99| = .005050, \quad |985\sqrt{2} - 1393| = .000358,$$

which are small when measured against $1/70 = .014285$ and $1/985 = .001015$. We see vividly that $99/70$ and $1393/985$ really should be judged as "good" rational approximations to $\sqrt{2}$ while the decimal truncations are "bad" rational approximations to $\sqrt{2}$.

The importance of this point of view is that it gives us a general *strategy* for proving numbers are irrational, as follows.

Theorem 4.1. *Let $\alpha \in \mathbf{R}$. If there is a sequence of integers p_n, q_n such that $q_n\alpha - p_n \neq 0$ and $|q_n\alpha - p_n| \to 0$ as $n \to \infty$, then α is irrational.*

In other words, if α admits a "very good" sequence of rational approximations, then α must be irrational.

Proof. Since $0 < |q_n\alpha - p_n| < 1$ for large n, by hypothesis, we must have $q_n \neq 0$ for large n. Therefore, since only large n is what matters, we may change terms at the start and assume $q_n \neq 0$ for all n.

To prove α is irrational, suppose it is rational: $\alpha = a/b$, where a and b are integers with $b \neq 0$. Then

$$\left|\alpha - \frac{p_n}{q_n}\right| = \left|\frac{a}{b} - \frac{p_n}{q_n}\right| = \left|\frac{q_n a - p_n b}{bq_n}\right|.$$

Clearing the denominator q_n,

$$|q_n\alpha - p_n| = \left|\frac{q_n a - p_n b}{b}\right|.$$

Since this is not zero, the integer $q_n a - p_n b$ is nonzero. Therefore $|q_n a - p_n b| \geq 1$, so

$$|q_n\alpha - p_n| \geq \frac{1}{|b|}.$$

This positive lower bound contradicts $|q_n\alpha - p_n|$ tending to 0, so α is irrational. □

It turns out the condition in Theorem 4.1 is not just sufficient to prove irrationality, but it is also necessary: if α is irrational then there is a sequence of integers p_n, q_n such that $|q_n\alpha - p_n|$ is not zero and tends to 0 as $n \to \infty$. We will not have a need for this necessity (except maybe for its psychological boost) and therefore omit the proof. See [4, p. 277].

Of course, to use Theorem 4.1 to prove irrationality of a number α we need to *find* the integers p_n and q_n. For the number e, these integers can be found directly from truncations to the infinite series for e, as we saw in (3.1). In other words, rather than saying e is irrational because the proof of Theorem 3.1 shows in the end that rationality of e leads to an integer between 0 and 1, we can say e is irrational because the proof of Theorem 3.1 exhibits a sequence of good rational approximations to e. In other words, the proof of Theorem 3.1 can stop at (3.1) and then appeal to Theorem 4.1.

While other powers of e are also irrational, it is not feasible to prove their irrationality by adapting the proof of Theorem 3.1. For instance, what happens if we try to prove e^2 is irrational from taking truncations of the infinite series $e^2 = \sum_{k\geq 0} 2^k/k!$? Writing the truncated sum $\sum_{k=0}^{n} 2^k/k!$ in reduced form as, say, a_n/b_n, numerical data suggest $b_n e^2 - a_n$ does *not* tend to 0, (For example, the value of $b_n e^2 - a_n$ at $n = 22, 23,$ and 24 is roughly $.0026, 1.4488,$ and $.3465$. Since the corresponding values of b_n have 12, 16, and 17 decimal digits, these differences are not small by comparison with $1/b_n$, so the approximations a_n/b_n to e^2 are not that good.) Thus, these rational approximations to e^2 probably won't fit the conditions of Theorem 4.1 to let us prove the irrationality of e^2. (However, a well-chosen *subsequence* of the partial sums does work. See the appendix.)

5. Irrationality of rational powers of e

To find good rational approximations to positive integral powers of e (good enough, that is, to establish irrationality of those powers), we will not use a series expansion, but rather use the interaction between e^x and integration. Some of the mysterious ideas from Niven's proof of the irrationality of π will show up in this context.

We will use Theorem 4.1 to prove the following generalization of the irrationality of e.

Theorem 5.1. *For every positive integer a, e^a is irrational.*

Before we prove Theorem 5.1, we note two immediate corollaries.

Corollary 5.2. *When r is a nonzero rational number, e^r is irrational.*

Proof. Since $e^{-r} = 1/e^r$, it suffices to take $r > 0$. Then $r = a/b$ for positive integers a and b. If e^r is rational, so is $(e^r)^b = e^a$, but this contradicts Theorem 5.1. Thus e^r is irrational. □

Corollary 5.3. *For each positive rational number $r \neq 1$, $\ln r$ is irrational.*

Proof. The number $\ln r$ is nonzero. If $\ln r$ is rational, then Corollary 5.2 tells us $e^{\ln r}$ is irrational. But $e^{\ln r} = r$ is rational. We have a contradiction, so $\ln r$ is irrational. □

The proof of Theorem 5.1 will use the following lemma, which tells us how to integrate $e^{-x} f(x)$ when $f(x)$ is a polynomial.

Lemma 5.4 (Hermite)**.** *Let $f(x)$ be a polynomial of degree $m \geq 0$. For every real number $a > 0$,*

$$\int_0^a e^{-x} f(x)\,\mathrm{d}x = \sum_{j=0}^{m} f^{(j)}(0) - e^{-a} \sum_{j=0}^{m} f^{(j)}(a).$$

Proof. We compute $\int e^{-x} f(x)\,\mathrm{d}x$ by integration by parts, taking $u = f(x)$ and $\mathrm{d}v = e^{-x}\mathrm{d}x$. Then $\mathrm{d}u = f'(x)\mathrm{d}x$ and $v = -e^{-x}$, so

$$\int e^{-x} f(x)\,\mathrm{d}x = -e^{-x} f(x) + \int e^{-x} f'(x)\,\mathrm{d}x.$$

Repeating this process on the new indefinite integral, we eventually obtain

$$\int e^{-x} f(x)\,\mathrm{d}x = -e^{-x} \sum_{j=0}^{m} f^{(j)}(x).$$

Now evaluate the right side at $x = a$ and $x = 0$ and subtract. □

Remark 5.5. It is interesting to make a special case of this lemma explicit: for $f(x) = x^n$,

$$\int_0^a e^{-x} x^n\,\mathrm{d}x = n! - \frac{1}{e^a} \sum_{j=0}^{n} n(n-1)\cdots(n-j+1)a^{n-j}.$$

Letting $a \to \infty$ (n is fixed), the second term on the right tends to 0, so $\int_0^\infty e^{-x} x^n\,\mathrm{d}x = n!$. This integral formula for $n!$ is due to Euler.

Now we prove Theorem 5.1.

Proof. Rewrite Hermite's lemma (Lemma 5.4) by multiplying through by e^a:

$$e^a \int_0^a e^{-x} f(x)\,\mathrm{d}x = e^a \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(a). \tag{5.1}$$

Equation (5.1) is valid for every positive number a and polynomial $f(x)$. Let a be a positive integer at which e^a is assumed to be rational. We want to use for $f(x)$ a polynomial (actually, a sequence of polynomials $f_n(x)$) with two properties:

- the left side of (5.1) is positive and (when n is large) very small,
- all the derivatives of the polynomial at $x = 0$ and $x = a$ are integers.

Then the right side of (5.1) will have the properties of the differences $q_n\alpha - p_n$ in Theorem 4.1, with $\alpha = e^a$ and the two sums on the right side of (5.1) being p_n and q_n.

Our choice of $f(x)$ is

$$f_n(x) = \frac{x^n(x-a)^n}{n!}, \tag{5.2}$$

where $n \geq 1$ is to be determined. (Note the similarity with (2.2) in the proof of the irrationality of π!) In other words, we consider the equation

$$e^a \int_0^a e^{-x} f_n(x)\,\mathrm{d}x = e^a \sum_{j=0}^{2n} f_n^{(j)}(0) - \sum_{j=0}^{2n} f_n^{(j)}(a). \tag{5.3}$$

We can see (5.3) is positive by looking at the left side. The number a is positive and the integrand $e^{-x}f_n(x) = e^{-x}x^n(x-a)^n/n!$ on the interval $(0,a)$ is positive, so the integral is positive. Now we estimate the size of (5.3) by estimating the integral on the left side. By the change of variables $x = ay$ on the left side of (5.3),

$$\int_0^a e^{-x} f_n(x)\,\mathrm{d}x = a^{2n+1} \int_0^1 e^{-ay} \frac{y^n(y-1)^n}{n!}\,\mathrm{d}y,$$

so we can bound the left side of (5.3) from above by

$$0 < e^a \int_0^a e^{-x} f_n(x)\,\mathrm{d}x \leq \frac{e^a a^{2n+1}}{n!} \int_0^1 e^{-ay}\,\mathrm{d}y.$$

As a function of n, this upper bound is a constant times $(a^2)^n/n!$. As $n \to \infty$, this bound tends to 0.

To see that, for each integer $n \geq 1$, the derivatives $f_n^{(j)}(0)$ and $f_n^{(j)}(a)$ are integers for every $j \geq 0$, first note that the equation $f_n(a-x) = f_n(x)$ tells us after repeated differentiation that $(-1)^j f_n^{(j)}(a) = f_n^{(j)}(0)$. Therefore it suffices to show all the derivatives of $f_n(x)$ at $x = 0$ are integers. The proof that all $f_n^{(j)}(0)$ are integers is just like that in the proof of Theorem 2.1, so the details are left to the reader to check. (The general principle is this: for a polynomial $g(x)$ that has integer coefficients and is divisible by x^n, all derivatives of $g(x)/n!$ at $x = 0$ are integers.)

The first property of the f_n's tells us that $q_ne^a - p_n$ is positive, where $p_n = \sum_{j=0}^{2n} f_n^{(j)}(a)$ and $q_n = \sum_{j=0}^{2n} f_n^{(j)}(a)$, and $q_ne^a - p_n$ tends to 0 as $n \to \infty$. The second property of the f_n's tells us that p_n and q_n are integers. Therefore the hypotheses of Theorem 4.1 are met, so e^a is irrational. □

What really happened in this proof? We actually wrote down some very good rational approximations to e^a. They came from values of the polynomial

$$F_n(x) = \sum_{j=0}^{2n} f_n^{(j)}(x).$$

Indeed, Theorem 5.1 tells us $F_n(a)/F_n(0)$ is a "good" rational approximation to e^a when n is large. (The dependence of $F_n(x)$ on a is hidden in the formula for $f_n(x)$.) The following table illustrates this for $a = 2$, where the entry at $n = 1$ is pretty bad since $F_1(0) = 0$.

n	$\lvert F_n(0)e^2 - F_n(2)\rvert$
1	4
2	1.5562
3	.43775
4	.09631
5	.01739
6	.00266
7	.00035
8	.00004

If we take $a = 1$, the rational approximations we get for $e^a = e$ by this method are *different* from the partial sums $\sum_{k=0}^{n} 1/k!$.

Although the proofs of Theorems 2.1 and 5.1 are similar in the sense that both used estimates on integrals, the proof of Theorem 2.1 did not show π is irrational by exhibiting a sequence of good rational approximations to π. The proof of Theorem 2.1 was an "integer between 0 and 1" proof by contradiction. No good rational approximations to π were produced in that proof. It is simply *harder* to get our grips on π than it is on powers of e.

6. Returning to irrationality of π

The proof of Theroem 5.1 gives a broader context for the proof in Theorem 2.1 that π is irrational. In fact, by thinking about Theorem 5.1 in the *complex plane*, we are led to a slightly different proof of Theorem 2.1.

Proof. We are going to use Hermite's lemma for complex a, where integrals from 0 to a are obtained using a path of integration between 0 and a (such as the straightline path). The definition of e^a for complex a is the infinite series $\sum_{n\geq 0} a^n/n!$. In particular, for real t, breaking up the series for e^{it} into its real and imaginary parts yields

$$e^{it} = \cos t + i \sin t.$$

Thus $|e^{it}| = 1$ and $e^{i\pi} = -1$. Consider Hermite's lemma at $a = i\pi$:

$$\int_0^{i\pi} e^{-x} f(x)\,\mathrm{d}x = \sum_{j=0}^{m} f^{(j)}(0) + \sum_{j=0}^{m} f^{(j)}(i\pi). \tag{6.1}$$

Extending (5.2) to the complex plane, we will use in (6.1) $f(x) = f_n(x) = x^n(x - i\pi)^n/n!$ for some large n to be determined later (so $m = 2n$ as before). To put (6.1) in a more appealing form, we apply the change of variables $x = i\pi y$. The left side of (6.1) becomes

$$\int_0^{i\pi} e^{-x} f_n(x)\,\mathrm{d}x = i(-1)^n \pi^{2n+1} \int_0^1 e^{-i\pi y} \frac{y^n(y-1)^n}{n!}\,\mathrm{d}y. \tag{6.2}$$

Let $g_n(y) = y^n(y-1)^n/n!$ (which does not involve i or π), so $f_n(i\pi y) = (i\pi)^{2n} g_n(y)$ and differentiating j times gives

$$(i\pi)^j f_n^{(j)}(i\pi y) = (i\pi)^{2n} g_n^{(j)}(y). \tag{6.3}$$

Feeding (6.2) and (6.3) into (6.1), with $f = f_n$,

$$i(-1)^n \pi^{2n+1} \int_0^1 e^{-i\pi y} g_n(y)\,\mathrm{d}y = \sum_{j=0}^{2n} (i\pi)^{2n-j} g_n^{(j)}(0) + \sum_{j=0}^{2n} (i\pi)^{2n-j} g_n^{(j)}(1). \tag{6.4}$$

This is the key equation in the proof. It serves the role for us now that (5.3) did in the proof of irrationality of powers of e. Notice the numbers $g_n^{(j)}(0)$ and $g_n^{(j)}(1)$ are all integers.

Suppose (at last) that π is rational, say with positive denominator q. For $j < n$, $g_n^{(j)}(x)$ vanishes at $x = 0$ and $x = 1$, so the sums in (6.4) really only need to start at $j = n$. That means the largest power of π in the (nonzero) terms on the right side of (6.4) is π^n, so the largest denominator on the right side of (6.4) is q^n. Multiply both sides by q^n:

$$i(-q)^n\pi^{2n+1}\int_0^1 e^{-i\pi y}g_n(y)\,\mathrm{d}y = \sum_{j=0}^{2n}(i\pi)^{2n-j}q^n g_n^{(j)}(0) + \sum_{j=0}^{2n}(i\pi)^{2n-j}q^n g_n^{(j)}(1). \tag{6.5}$$

We estimate the left side of (6.5). Since $|e^{i\pi y}| = 1$, an estimate of the left side is

$$\left|q^n\pi^{2n+1}\int_0^1 e^{-i\pi y}g_n(y)\,\mathrm{d}y\right| \le \frac{q^n\pi^{2n+1}}{n!} = \pi\frac{(q\pi^2)^n}{n!}.$$

As $n \to \infty$, this bound tends to 0.

On the other hand, since the nonzero terms in the sums on the right side of (6.5) only start showing up at the $j = n$ term, and $\pi^{2n-j}q^n \in \mathbf{Z}$ for $n \le j \le 2n$, the right side of (6.5) is in the integral lattice $\mathbf{Z} + \mathbf{Z}i$. It is nonzero, as we see by looking at the real part of the left side of (6.5), which is

$$(-q)^n\pi^{2n+1}\int_0^1 \sin(\pi y)g_n(y)\,\mathrm{d}y = (-q)^n\pi^{2n+1}\int_0^1 \sin(\pi y)\frac{y^n(y-1)^n}{n!}\,\mathrm{d}y.$$

The integrand here has constant sign on $(0,1)$, so (6.5) is in $\mathbf{Z} + \mathbf{Z}i$ and doesn't vanish for integers $n \ge 1$ since the real part is not 0. A sequence of nonzero elements of $\mathbf{Z} + \mathbf{Z}i$ can't tend to 0. We have a contradiction, so π is irrational. $\square$

We used complex numbers in the above proof to stress the close connection to the proof of Theorem 5.1. If you take the real part of every equation in the above proof (especially starting with (6.4)), then you will find a proof of the irrationality of π that avoids complex numbers. (For instance, we showed (6.5) is nonzero by looking only at the real part, so taking real parts everywhere should not damage the logic of the proof.) By taking real parts, the goal of the proof changes slightly. Instead of showing the rationality of π leads to a nonzero element of $\mathbf{Z}+\mathbf{Z}i$ with absolute value less than 1, showing the rationality of π leads to a nonzero integer with absolute value less than 1. Is such a "real" proof basically the same as the first proof we gave that π is rational? As a check, try to adapt the first proof of Theorem 2.1 to get the following.

Corollary 6.1. *The number π^2 is irrational.*

Proof. Run through the previous proof, starting at (6.4), but now assume π^2 is rational with denominator $q \in \mathbf{Z}$. While it is no longer true that $\pi^{2n-j}q^n \in \mathbf{Z}$ for $n \le j \le 2n$, we instead have $\pi^{2n-j}q^n \in \mathbf{Z}$ for j even and $\pi^{2n-j}q^n \in (1/\pi)\mathbf{Z}$ for j odd. Since i^{2n-j} is real when j is even and imaginary when j is odd, we now have (6.5) lying in the set $\mathbf{Z}+\mathbf{Z}(i/\pi)$, whose nonzero elements are not arbitrarily small. $\square$

If you write up all the details in this proof and take the real part of every equation, you will have the proof of the irrationality of π in [7, Chap. 16].

The numbers π and e are not just irrational, but transcendental. That is, neither number is the root of a nonzero polynomial with rational coefficients. (For comparison, $\sqrt{2}$ is irrational but it is a solution of $x^2 - 2 = 0$, so it is in some sense linked to the rational numbers through this equation.) Proofs of their transcendence can be found in [1, Chap. 1], [2, Chap. II], and [6, Chap. 2]. The proof that e is transcendental is actually not much more complicated than the our proof of Theorem 5.1, taking perhaps two pages rather than one. The idea is to use Hermite's lemma and a construction of a sequence of rational functions whose values give good rational approximations *simultaneously* to several integral powers of e, not only to one power like e^a. The construction of such rational functions generalizes the $f_n(x)$'s in Theorem 5.1, which gave good rational approximations to e^a. By comparison to this, the proof that π is transcendental is much more involved than the proof of its irrationality.

Historically, progress on π always trailed that of e. Euler proved e is irrational in 1737,and Lambert proved irrationality of non-zero rational powers of e and irrationality of π in 1761. Their proofs used continued fractions, not integrals. (Lambert's proof for π was actually a result about the tangent function. When r is a nonzero rational where the tangent function is defined, Lambert proved $\tan r$ is irrational. Then, since $\tan(\pi/4) = 1$ is rational, π must be irrational in order to avoid a contradiction.) Transcendence proofs for e and π came 100 years later, in the work of Hermite (1873) and Lindemann (1882). In addition to e and π, the numbers $2^{\sqrt{2}}$, $\log 2$, and e^π are known to be transcendental. The status of 2^e, π^e, $\pi^{\sqrt{2}}$, $e+\pi$, $e\log 2$, and Euler's constant $\gamma = \lim_{n\to\infty}\sum_{k=1}^n 1/k - \log n$ is still open. Surely these numbers are all transcendental, but it is not yet proved that even one of them is irrational.

Appendix A. Irrationality of e^2

In Section 4, we saw that the partial sums of the Taylor series for e^x at $x = 2$, namely the sums $\sum_{k=0}^n 2^k/k!$, do not seem to be a sequence of rational approximations to e^2 that allow us to prove e^2 is irrational via Theorem 4.1. This was circumvented in Theorem 5.1, where the nonzero integral powers of e (not just e^2) were proved to be irrational using rational approximations coming from something other than the Taylor series for e^x. What we will show here is that the partial sums $\sum_{k=0}^n 2^k/k!$ can, after all, be used to prove irrationality of e^2 by focusing on a certain subsequence of the partial sums and exploiting a peculiar property of 2. The argument we give is due to Benoit Cloitre.

Write $\sum_{k=0}^n 2^k/k!$ in reduced form as a_n/b_n. Then $e^2 > a_n/b_n$, so $b_ne^2 - a_n > 0$. While numerical data suggest $b_ne^2-a_n$ does not go to 0 as $n\to\infty$, it turns out that these nonzero differences tend to 0 as n runs through the powers of 2. The table below has some limited evidence in this direction.

n	$b_ne^2 - a_n$
2	2.389
4	.3890
8	.5526
16	.0881
32	.0006
64	.0211
128	.0005
256	.0001

If these differences do tend to 0, then this proves e^2 is irrational by Theorem 4.1. To prove this phenomenon is real, we use Lagrange's form of the remainder to estimate the difference between e^2 and $\sum_{k=0}^n 2^k/k!$ before setting n to be a power of 2. Lagrange's form of the remainder says: for an infinitely differentiable function f and integer $n \geq 0$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(c)}{(n+1)!}x^n,$$

where c is between 0 and x. Taking f to be the exponential function and $x = 2$,

$$e^2 = \sum_{k=0}^{n} \frac{2^k}{k!} + \frac{e^c 2^n}{(n+1)!},$$

where $0 < c < 2$. Bring the sum to the left side, multiply by $n!/2^n$, and take absolute values:

$$\left|\frac{n!}{2^n}e^2 - \frac{n!}{2^n}\sum_{k=0}^{n}\frac{2^k}{k!}\right| \leq \frac{e^2}{n+1}. \tag{A.1}$$

Set

$$c_n = \frac{n!}{2^n}\sum_{k=0}^{n}\frac{2^k}{k!}, \quad d_n = \frac{n!}{2^n},$$

so $c_n/d_n = \sum_{k=0}^{n} 2^k/k! = a_n/b_n$ and (A.1) says $|d_n e^2 - c_n| \leq e^2/(n+1)$. Thus $|d_n e^2 - c_n| \to 0$ as $n \to \infty$ and $d_n e^2 - c_n \neq 0$. Be careful: the numbers c_n and d_n are not themselves integers (as we will see), so the expression $|d_n e^2 - c_n|$ is not quite in a form suitable for immediate application of Theorem 4.1. But we can get good control on the denominators of c_n and d_n.

Writing $c_n = (1/2^n)\sum_{k=0}^{n}(n!/k!)2^k$, since $n!/k!$ is an integer c_n is an integer divided by 2^n, so c_n has a 2-power denominator in reduced form. Since d_n is an integer divided by 2^n, its reduced form denominator is also a power of 2. What are the powers of 2 in the reduced form for c_n and d_n?

For a nonzero rational number r, write $\mathrm{ord}_2(r)$ for the power of 2 appearing in r, *e.g.*, $\mathrm{ord}_2(40) = 3$ and $\mathrm{ord}_2(21/20) = -2$. By unique prime factorization, $\mathrm{ord}_2(rr') = \mathrm{ord}_2(r) + \mathrm{ord}_2(r')$ for nonzero rationals r and r'. Another important formula is $\mathrm{ord}_2(r+r') = \min(\mathrm{ord}_2(r), \mathrm{ord}_2(r'))$ when $\mathrm{ord}_2(r) \neq \mathrm{ord}_2(r')$ and $r + r' \neq 0$. (These properties both resemble the degree on polynomials and rational functions, except the degree has a max where ord_2 has a min on sums.)

We want to compute $\mathrm{ord}_2(c_n)$ and $\mathrm{ord}_2(d_n)$. As $d_n = n!/2^n$ is simpler than c_n, we look at it first. Since $\mathrm{ord}_2(d_n) = \mathrm{ord}_2(n!/2^n) = \mathrm{ord}_2(n!) - n$, we bring in a formula for the highest power of 2 in a factorial, due to Legendre: $\mathrm{ord}_2(n!) = n - s_2(n)$, where $s_2(n)$ is the sum of the base 2 digits of n. For example, $6! = 2^4 \cdot 3^2 \cdot 5$ and $6 = 2 + 2^2$, so $6 - s_2(6) = 6 - 2 = 4$ matches $\mathrm{ord}_2(6!)$. With this formula of Legendre,

$$\mathrm{ord}_2(d_n) = \mathrm{ord}_2(n!) - n = (n - s_2(n)) - n = -s_2(n).$$

For $n \geq 1$ this is negative (there is at least one nonzero base 2 digit in n, so $s_2(n) \geq 1$), which proves d_n is not an integer.

What about $\mathrm{ord}_2(c_n)$? Writing

$$c_n = \sum_{k=0}^{n} \frac{n!2^k}{2^n k!} = \frac{n!}{2^n} + \frac{n! \cdot 2}{2^n \cdot 1} + \frac{n! \cdot 4}{2^n \cdot 2} + \frac{n! \cdot 2^3}{2^n \cdot 6} + \cdots + 1,$$

each term has at worst a 2-power denominator since $n!/k! \in \mathbf{Z}$ when $0 \leq k \leq n$. To figure out the power of 2 in the denominator we will compute $\mathrm{ord}_2(n!2^k/2^n k!)$. For $k = 0$ this is $\mathrm{ord}_2(n!/2^n) = -s_2(n)$. For $k \geq 1$ this is

$$\mathrm{ord}_2(n!) + k - n - \mathrm{ord}_2(k!) = -s_2(n) + s_2(k) > -s_2(n)$$

since $s_2(k) \geq 1$. Therefore every term in the sum for c_n beyond the $k = 0$ term has a larger 2-power divisibility than the $k = 0$ term, which means $\mathrm{ord}_2(c_n)$ is the same as $\mathrm{ord}_2(n!/2^n) = -s_2(n)$. In other words, c_n and d_n have the same denominator: $2^{s_2(n)}$.

If we let n be a power of 2 then c_n and d_n have denominator $2^1 = 2$, so $2c_n$ and $2d_n$ are integers. Then the estimate

$$0 < |2d_n e^2 - 2c_n| = 2|d_n e^2 - c_n| \leq \frac{2e^2}{n+1} \to 0$$

proves e^2 is irrational by Theorem 4.1.

It is natural to ask if the same argument yields a proof that e^p is irrational for prime $p > 2$ using the Taylor series for e^x at $x = p$. Let $c_n = (n!/p^n)\sum_{k=0}^{n} p^k/k!$ and $d_n = n!/p^n$, so $|d_n e^p - c_n| \leq e^p/(n+1)$ as before. Legendre's formula for $\mathrm{ord}_p(n!)$, the highest power of p in $n!$, is $(n - s_p(n))/(p-1)$, where $s_p(n)$ is the sum of the base p digits of n. The fractions c_n and d_n have the same p-power denominator, with exponent $\mathrm{ord}_p(n!) - n = n(1 - 1/(p-1)) + s_p(n)/(p-1)$. Alas, for $p > 2$ this exponent of p in the denominator of c_n and d_n blows up with n because $1 - 1/(p-1) > 0$ for $p > 2$. When $p = 2$ this exponent in the denominator is $s_2(n)$, which can stay bounded by restricting n to the powers of 2.

References

[1] A. Baker, "Transcendental Number Theory," Cambridge Univ. Press, Cambridge, 1975.
[2] A. O. Gelfond, "Transcendental and Algebraic Numbers," Dover, New York, 1960.
[3] E. Hairer and G. Wanner, "Analysis by its History," Springer-Verlag, New York, 1996.
[4] K. Ireland and M. Rosen, "A Classical Introduction to Modern Number Theory," 2nd ed., Springer-Verlag, New York, 1990.
[5] I. Niven, A simple proof that π is irrational, *Bull. Amer. Math. Soc.* **53** (1947), 509.
[6] A. Shidlovski, "Transcendental Numbers," de Gruyter, Berlin 1989.
[7] M. Spivak, "Calculus," 2nd ed., Publish or Perish, Wilmington, Del., 1967.
[8] L. Zhou and L. Markov, Recurrent Proofs of the Irrationality of Certain Trigonometric Values, *Amer. Math. Monthly* **117** (2010), 360–362.