## THE GAUSSIAN INTEGRAL

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Let

$$
I=\int_{-\infty}^{\infty} e^{-\frac{1}{2} x^{2}} \mathrm{~d} x, J=\int_{0}^{\infty} e^{-x^{2}} \mathrm{~d} x, \quad \text { and } K=\int_{-\infty}^{\infty} e^{-\pi x^{2}} \mathrm{~d} x
$$

These positive numbers are related: $J=I /(2 \sqrt{2})$ and $K=I / \sqrt{2 \pi}$.
Theorem. With notation as above, $I=\sqrt{2 \pi}$, or equivalently $J=\sqrt{\pi} / 2$, or equivalently $K=1$.
We will give multiple proofs of this. (Other lists of proofs are in [4] and [9].) It is subtle since $e^{-\frac{1}{2} x^{2}}$ has no simple antiderivative. For comparison, $\int_{0}^{\infty} x e^{-\frac{1}{2} x^{2}} \mathrm{~d} x$ can be computed with the antiderivative $-e^{-\frac{1}{2} x^{2}}$ and equals 1 . In the last section, the Gaussian integral's history is presented.

## 1. First Proof: Polar coordinates

The most widely known proof, due to Poisson [9, p. 3], expresses $J^{2}$ as a double integral and then uses polar coordinates. To start, write $J^{2}$ as an iterated integral using single-variable calculus:

$$
J^{2}=J \int_{0}^{\infty} e^{-y^{2}} \mathrm{~d} y=\int_{0}^{\infty} J e^{-y^{2}} \mathrm{~d} y=\int_{0}^{\infty}\left(\int_{0}^{\infty} e^{-x^{2}} \mathrm{~d} x\right) e^{-y^{2}} \mathrm{~d} y=\int_{0}^{\infty} \int_{0}^{\infty} e^{-\left(x^{2}+y^{2}\right)} \mathrm{d} x \mathrm{~d} y
$$

View this as a double integral over the first quadrant. To compute it with polar coordinates, the first quadrant is $\{(r, \theta): r \geq 0$ and $0 \leq \theta \leq \pi / 2\}$. Writing $x^{2}+y^{2}$ as $r^{2}$ and $\mathrm{d} x \mathrm{~d} y$ as $r \mathrm{~d} r \mathrm{~d} \theta$,

$$
\begin{aligned}
J^{2} & =\int_{0}^{\pi / 2} \int_{0}^{\infty} e^{-r^{2}} r \mathrm{~d} r \mathrm{~d} \theta \\
& =\int_{0}^{\infty} r e^{-r^{2}} \mathrm{~d} r \cdot \int_{0}^{\pi / 2} \mathrm{~d} \theta \\
& =-\left.\frac{1}{2} e^{-r^{2}}\right|_{0} ^{\infty} \cdot \frac{\pi}{2} \\
& =\frac{1}{2} \cdot \frac{\pi}{2} \\
& =\frac{\pi}{4} .
\end{aligned}
$$

Since $J>0, J=\sqrt{\pi} / 2 \cdot{ }^{1}$ It is argued in [1] that this method can't be used on any other integral.

## 2. Second Proof: Another change of variables

Our next proof uses another change of variables to compute $J^{2}$. As before,

$$
J^{2}=\int_{0}^{\infty}\left(\int_{0}^{\infty} e^{-\left(x^{2}+y^{2}\right)} \mathrm{d} x\right) \mathrm{d} y
$$

[^0]Instead of using polar coordinates, set $x=y t$ in the inner integral ( $y$ is fixed). Then $\mathrm{d} x=y \mathrm{~d} t$ and

$$
\begin{equation*}
J^{2}=\int_{0}^{\infty}\left(\int_{0}^{\infty} e^{-y^{2}\left(t^{2}+1\right)} y \mathrm{~d} t\right) \mathrm{d} y=\int_{0}^{\infty}\left(\int_{0}^{\infty} y e^{-y^{2}\left(t^{2}+1\right)} \mathrm{d} y\right) \mathrm{d} t \tag{2.1}
\end{equation*}
$$

where the interchange of integrals is justified by Fubini's theorem for improper Riemann integrals. (The appendix gives an approach using Fubini's theorem for Riemann integrals on rectangles.) Since $\int_{0}^{\infty} y e^{-a y^{2}} \mathrm{~d} y=\frac{1}{2 a}$ for $a>0$, we have

$$
J^{2}=\int_{0}^{\infty} \frac{\mathrm{d} t}{2\left(t^{2}+1\right)}=\frac{1}{2} \cdot \frac{\pi}{2}=\frac{\pi}{4},
$$

so $J=\sqrt{\pi} / 2$. This proof is due to Laplace [7, pp. 94-96] and historically precedes the widely used technique of the previous proof. We will see in Section 9 what Laplace's first proof was.

## 3. Third Proof: Differentiating under the integral sign

For $t>0$, set

$$
A(t)=\left(\int_{0}^{t} e^{-x^{2}} \mathrm{~d} x\right)^{2}
$$

The integral we want to calculate is $A(\infty)=J^{2}$ and then take a square root.
Differentiating $A(t)$ with respect to $t$ and using the Fundamental Theorem of Calculus,

$$
A^{\prime}(t)=2 \int_{0}^{t} e^{-x^{2}} \mathrm{~d} x \cdot e^{-t^{2}}=2 e^{-t^{2}} \int_{0}^{t} e^{-x^{2}} \mathrm{~d} x
$$

Let $x=t y$, so

$$
A^{\prime}(t)=2 e^{-t^{2}} \int_{0}^{1} t e^{-t^{2} y^{2}} \mathrm{~d} y=\int_{0}^{1} 2 t e^{-\left(1+y^{2}\right) t^{2}} \mathrm{~d} y
$$

The function under the integral sign is easily antidifferentiated with respect to $t$ :

$$
A^{\prime}(t)=\int_{0}^{1}-\frac{\partial}{\partial t} \frac{e^{-\left(1+y^{2}\right) t^{2}}}{1+y^{2}} \mathrm{~d} y=-\frac{d}{d t} \int_{0}^{1} \frac{e^{-\left(1+y^{2}\right) t^{2}}}{1+y^{2}} \mathrm{~d} y .
$$

Letting

$$
B(t)=\int_{0}^{1} \frac{e^{-t^{2}\left(1+x^{2}\right)}}{1+x^{2}} \mathrm{~d} x
$$

we have $A^{\prime}(t)=-B^{\prime}(t)$ for all $t>0$, so there is a constant $C$ such that

$$
\begin{equation*}
A(t)=-B(t)+C \tag{3.1}
\end{equation*}
$$

for all $t>0$. To find $C$, we let $t \rightarrow 0^{+}$in (3.1). The left side tends to $\left(\int_{0}^{0} e^{-x^{2}} \mathrm{~d} x\right)^{2}=0$ while the right side tends to $-\int_{0}^{1} \mathrm{~d} x /\left(1+x^{2}\right)+C=-\pi / 4+C$. Thus $C=\pi / 4$, so (3.1) becomes

$$
\left(\int_{0}^{t} e^{-x^{2}} \mathrm{~d} x\right)^{2}=\frac{\pi}{4}-\int_{0}^{1} \frac{e^{-t^{2}\left(1+x^{2}\right)}}{1+x^{2}} \mathrm{~d} x .
$$

Letting $t \rightarrow \infty$ in this equation, we obtain $J^{2}=\pi / 4$, so $J=\sqrt{\pi} / 2$.
A comparison of this proof with the first proof is in [21].

## 4. Fourth Proof: Another differentiation under the integral sign

Here is a second approach to finding $J$ by differentiation under the integral sign. I heard about it from Michael Rozman [14], who modified an idea on math. stackexchange [23], and in a slightly less elegant form it appeared much earlier in [19].

For $t \in \mathbf{R}$, set

$$
F(t)=\int_{0}^{\infty} \frac{e^{-t^{2}\left(1+x^{2}\right)}}{1+x^{2}} \mathrm{~d} x
$$

Then $F(0)=\int_{0}^{\infty} d x /\left(1+x^{2}\right)=\pi / 2$ and $F(\infty)=0$. Differentiating under the integral sign,

$$
F^{\prime}(t)=\int_{0}^{\infty}-2 t e^{-t^{2}\left(1+x^{2}\right)} \mathrm{d} x=-2 t e^{-t^{2}} \int_{0}^{\infty} e^{-(t x)^{2}} \mathrm{~d} x
$$

Make the substitution $y=t x$, with $\mathrm{d} y=t \mathrm{~d} x$, so

$$
F^{\prime}(t)=-2 e^{-t^{2}} \int_{0}^{\infty} e^{-y^{2}} \mathrm{~d} y=-2 J e^{-t^{2}}
$$

For $b>0$, integrate both sides from 0 to $b$ and use the Fundamental Theorem of Calculus:

$$
\int_{0}^{b} F^{\prime}(t) \mathrm{d} t=-2 J \int_{0}^{b} e^{-t^{2}} \mathrm{~d} t \Longrightarrow F(b)-F(0)=-2 J \int_{0}^{b} e^{-t^{2}} \mathrm{~d} t .
$$

Letting $b \rightarrow \infty$ in the last equation,

$$
0-\frac{\pi}{2}=-2 J^{2} \Longrightarrow J^{2}=\frac{\pi}{4} \Longrightarrow J=\frac{\sqrt{\pi}}{2}
$$

## 5. Fifth Proof: A volume integral

Our next proof is due to T. P. Jameson [5] and it was rediscovered by A. L. Delgado [3]. Revolve the curve $z=e^{-\frac{1}{2} x^{2}}$ in the $x z$-plane around the $z$-axis to produce the "bell surface" $z=e^{-\frac{1}{2}\left(x^{2}+y^{2}\right)}$. See below, where the $z$-axis is vertical and passes through the top point, the $x$-axis lies just under the surface through the point 0 in front, and the $y$-axis lies just under the surface through the point 0 on the left. We will compute the volume $V$ below the surface and above the $x y$-plane in two ways.

First we compute $V$ by horizontal slices, which are discs: $V=\int_{0}^{1} A(z) \mathrm{d} z$ where $A(z)$ is the area of the disc formed by slicing the surface at height $z$. Writing the radius of the disc at height $z$ as $r(z), A(z)=\pi r(z)^{2}$. To compute $r(z)$, the surface cuts the $x z$-plane at a pair of points $\left(x, e^{-\frac{1}{2} x^{2}}\right)$ where the height is $z$, so $e^{-\frac{1}{2} x^{2}}=z$. Thus $x^{2}=-2 \ln z$. Since $x$ is the distance of these points from the $z$-axis, $r(z)^{2}=x^{2}=-2 \ln z$, so $A(z)=\pi r(z)^{2}=-2 \pi \ln z$. Therefore

$$
V=\int_{0}^{1}-2 \pi \ln z \mathrm{~d} z=-\left.2 \pi(z \ln z-z)\right|_{0} ^{1}=-2 \pi\left(-1-\lim _{z \rightarrow 0^{+}} z \ln z\right) .
$$

By L'Hospital's rule, $\lim _{z \rightarrow 0^{+}} z \ln z=0$, so $V=2 \pi$. (A calculation of $V$ by shells is in [11].)
Next we compute the volume by vertical slices in planes $x=$ constant. Vertical slices are scaled bell curves: look at the black contour lines in the picture. The equation of the bell curve along the top of the vertical slice with $x$-coordinate $x$ is $z=e^{-\frac{1}{2}\left(x^{2}+y^{2}\right)}$, where $y$ varies and $x$ is fixed. Then

$V=\int_{-\infty}^{\infty} A(x) \mathrm{d} x$, where $A(x)$ is the area of the $x$-slice:

$$
A(x)=\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^{2}+y^{2}\right)} \mathrm{d} y=e^{-\frac{1}{2} x^{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} y^{2}} \mathrm{~d} y=e^{-\frac{1}{2} x^{2}} I
$$

Thus $V=\int_{-\infty}^{\infty} A(x) \mathrm{d} x=\int_{-\infty}^{\infty} e^{-\frac{1}{2} x^{2}} I \mathrm{~d} x=I \int_{-\infty}^{\infty} e^{-\frac{1}{2} x^{2}} \mathrm{~d} x=I^{2}$.
Comparing the two formulas for $V$, we have $2 \pi=I^{2}$, so $I=\sqrt{2 \pi}$.

## 6. Sixth Proof: The $\Gamma$-function

For any integer $n \geq 0$, we have $n!=\int_{0}^{\infty} t^{n} e^{-t} \mathrm{~d} t$. For $x>0$ we define

$$
\Gamma(x)=\int_{0}^{\infty} t^{x} e^{-t} \frac{\mathrm{~d} t}{t},
$$

so $\Gamma(n)=(n-1)$ ! when $n \geq 1$. Using integration by parts, $\Gamma(x+1)=x \Gamma(x)$. One of the basic properties of the $\Gamma$-function [15, pp. 193-194] is

$$
\begin{equation*}
\frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}=\int_{0}^{1} t^{x-1}(1-t)^{y-1} \mathrm{~d} t \tag{6.1}
\end{equation*}
$$

Set $x=y=1 / 2$ :

$$
\Gamma\left(\frac{1}{2}\right)^{2}=\int_{0}^{1} \frac{\mathrm{~d} t}{\sqrt{t(1-t)}}
$$

Note

$$
\Gamma\left(\frac{1}{2}\right)=\int_{0}^{\infty} \sqrt{t} e^{-t} \frac{\mathrm{~d} t}{t}=\int_{0}^{\infty} \frac{e^{-t}}{\sqrt{t}} \mathrm{~d} t=\int_{0}^{\infty} \frac{e^{-x^{2}}}{x} 2 x \mathrm{~d} x=2 \int_{0}^{\infty} e^{-x^{2}} \mathrm{~d} x=2 J,
$$

so $4 J^{2}=\int_{0}^{1} \mathrm{~d} t / \sqrt{t(1-t)}$. With the substitution $t=\sin ^{2} \theta$,

$$
4 J^{2}=\int_{0}^{\pi / 2} \frac{2 \sin \theta \cos \theta \mathrm{~d} \theta}{\sin \theta \cos \theta}=2 \frac{\pi}{2}=\pi
$$

so $J=\sqrt{\pi} / 2$. Equivalently, $\Gamma(1 / 2)=\sqrt{\pi}$. Any method that proves $\Gamma(1 / 2)=\sqrt{\pi}$ is also a method that calculates $\int_{0}^{\infty} e^{-x^{2}} \mathrm{~d} x$.

## 7. Seventh Proof: Asymptotic estimates

We will show $J=\sqrt{\pi} / 2$ by a technique whose steps are based on [16, p. 371].
For $x \geq 0$, power series expansions show $1+x \leq e^{x} \leq 1 /(1-x)$. Reciprocating and replacing $x$ with $x^{2}$, we get

$$
\begin{equation*}
1-x^{2} \leq e^{-x^{2}} \leq \frac{1}{1+x^{2}} \tag{7.1}
\end{equation*}
$$

for all $x \in \mathbf{R}$.
For any positive integer $n$, raise the terms in (7.1) to the $n$th power and integrate from 0 to 1 :

$$
\int_{0}^{1}\left(1-x^{2}\right)^{n} \mathrm{~d} x \leq \int_{0}^{1} e^{-n x^{2}} \mathrm{~d} x \leq \int_{0}^{1} \frac{\mathrm{~d} x}{\left(1+x^{2}\right)^{n}}
$$

Using the changes of variables $x=\sin \theta$ on the left, $x=y / \sqrt{n}$ in the middle, and $x=\tan \theta$ on the right,

$$
\begin{equation*}
\int_{0}^{\pi / 2}(\cos \theta)^{2 n+1} \mathrm{~d} \theta \leq \frac{1}{\sqrt{n}} \int_{0}^{\sqrt{n}} e^{-y^{2}} \mathrm{~d} y \leq \int_{0}^{\pi / 4}(\cos \theta)^{2 n-2} \mathrm{~d} \theta<\int_{0}^{\pi / 2}(\cos \theta)^{2 n-2} \mathrm{~d} \theta \tag{7.2}
\end{equation*}
$$

Set $I_{k}=\int_{0}^{\pi / 2}(\cos \theta)^{k} \mathrm{~d} \theta$, so $I_{0}=\pi / 2, I_{1}=1$, and (7.2) implies

$$
\begin{equation*}
\sqrt{n} I_{2 n+1} \leq \int_{0}^{\sqrt{n}} e^{-y^{2}} \mathrm{~d} y<\sqrt{n} I_{2 n-2} \tag{7.3}
\end{equation*}
$$

We will show that as $k \rightarrow \infty, k I_{k}^{2} \rightarrow \pi / 2$. Then

$$
\sqrt{n} I_{2 n+1}=\frac{\sqrt{n}}{\sqrt{2 n+1}} \sqrt{2 n+1} I_{2 n+1} \rightarrow \frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{2}}=\frac{\sqrt{\pi}}{2}
$$

and

$$
\sqrt{n} I_{2 n-2}=\frac{\sqrt{n}}{\sqrt{2 n-2}} \sqrt{2 n-2} I_{2 n-2} \rightarrow \frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{2}}=\frac{\sqrt{\pi}}{2}
$$

so by (7.3), $\int_{0}^{\sqrt{n}} e^{-y^{2}} \mathrm{~d} y \rightarrow \sqrt{\pi} / 2$. Thus $J=\sqrt{\pi} / 2$.

To show $k I_{k}^{2} \rightarrow \pi / 2$, first we compute several values of $I_{k}$ explicitly by a recursion. Using integration by parts,

$$
I_{k}=\int_{0}^{\pi / 2}(\cos \theta)^{k} \mathrm{~d} \theta=\int_{0}^{\pi / 2}(\cos \theta)^{k-1} \cos \theta \mathrm{~d} \theta=(k-1)\left(I_{k-2}-I_{k}\right),
$$

so

$$
\begin{equation*}
I_{k}=\frac{k-1}{k} I_{k-2} . \tag{7.4}
\end{equation*}
$$

Using (7.4) and the initial values $I_{0}=\pi / 2$ and $I_{1}=1$, the first few values of $I_{k}$ are computed and listed in Table 1.

| $k$ | $I_{k}$ | $k$ | $I_{k}$ |
| :---: | :---: | :---: | :---: |
| 0 | $\pi / 2$ | 1 | 1 |
| 2 | $(1 / 2)(\pi / 2)$ | 3 | $2 / 3$ |
| 4 | $(3 / 8)(\pi / 2)$ | 5 | $8 / 15$ |
| 6 | $(15 / 48)(\pi / 2)$ | 7 | $48 / 105$ |

Table 1.

From Table 1 we see that

$$
\begin{equation*}
I_{2 n} I_{2 n+1}=\frac{1}{2 n+1} \frac{\pi}{2} \tag{7.5}
\end{equation*}
$$

for $0 \leq n \leq 3$, and this can be proved for all $n$ by induction using (7.4). Since $0 \leq \cos \theta \leq 1$ for $\theta \in[0, \pi / 2]$, we have $I_{k} \leq I_{k-1} \leq I_{k-2}=\frac{k}{k-1} I_{k}$ by (7.4), so $I_{k-1} \sim I_{k}$ as $k \rightarrow \infty$. Therefore (7.5) implies

$$
I_{2 n}^{2} \sim \frac{1}{2 n} \frac{\pi}{2} \Longrightarrow(2 n) I_{2 n}^{2} \rightarrow \frac{\pi}{2}
$$

as $n \rightarrow \infty$. Then

$$
(2 n+1) I_{2 n+1}^{2} \sim(2 n) I_{2 n}^{2} \rightarrow \frac{\pi}{2}
$$

as $n \rightarrow \infty$, so $k I_{k}^{2} \rightarrow \pi / 2$ as $k \rightarrow \infty$. This completes our proof that $J=\sqrt{\pi} / 2$.
Remark 7.1. This proof is closely related to the fifth proof using the $\Gamma$-function. Indeed, by (6.1)

$$
\frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+1}{2}+\frac{1}{2}\right)}=\int_{0}^{1} t^{(k+1) / 2+1}(1-t)^{1 / 2-1} \mathrm{~d} t
$$

and with the change of variables $t=(\cos \theta)^{2}$ for $0 \leq \theta \leq \pi / 2$, the integral on the right is equal to $2 \int_{0}^{\pi / 2}(\cos \theta)^{k} \mathrm{~d} \theta=2 I_{k}$, so (7.5) is the same as

$$
\begin{aligned}
I_{2 n} I_{2 n+1} & =\frac{\Gamma\left(\frac{2 n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2 \Gamma\left(\frac{2 n+2}{2}\right)} \frac{\Gamma\left(\frac{2 n+2}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2 \Gamma\left(\frac{2 n+3}{2}\right)} \\
& =\frac{\Gamma\left(\frac{2 n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)^{2}}{4 \Gamma\left(\frac{2 n+1}{2}+1\right)} \\
& =\frac{\Gamma\left(\frac{2 n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)^{2}}{4 \frac{2 n+1}{2} \Gamma\left(\frac{2 n+1}{2}\right)} \\
& =\frac{\Gamma\left(\frac{1}{2}\right)^{2}}{2(2 n+1)} .
\end{aligned}
$$

By (7.5), $\pi=\Gamma(1 / 2)^{2}$. We saw in the fifth proof that $\Gamma(1 / 2)=\sqrt{\pi}$ if and only if $J=\sqrt{\pi} / 2$.

## 8. Eighth Proof: Stirling's Formula

Besides the integral formula $\int_{-\infty}^{\infty} e^{-\frac{1}{2} x^{2}} \mathrm{~d} x=\sqrt{2 \pi}$ that we have been discussing, another place in mathematics where $\sqrt{2 \pi}$ appears is in Stirling's formula:

$$
n!\sim \frac{n^{n}}{e^{n}} \sqrt{2 \pi n} \text { as } n \rightarrow \infty
$$

In 1730 De Moivre proved $n!\sim C\left(n^{n} / e^{n}\right) \sqrt{n}$ for some positive number $C$ without being able to determine $C$. Stirling soon thereafter showed $C=\sqrt{2 \pi}$ and wound up having the whole formula named after him. We will show that determining that the constant $C$ in Stirling's formula is $\sqrt{2 \pi}$ is equivalent to showing that $J=\sqrt{\pi} / 2$ (or, equivalently, that $I=\sqrt{2 \pi}$ ).

Applying (7.4) repeatedly,

$$
\begin{aligned}
I_{2 n} & =\frac{2 n-1}{2 n} I_{2 n-2} \\
& =\frac{(2 n-1)(2 n-3)}{(2 n)(2 n-2)} I_{2 n-4} \\
& \vdots \\
& =\frac{(2 n-1)(2 n-3)(2 n-5) \cdots(5)(3)(1)}{(2 n)(2 n-2)(2 n-4) \cdots(6)(4)(2)} I_{0} .
\end{aligned}
$$

Inserting $(2 n-2)(2 n-4)(2 n-6) \cdots(6)(4)(2)$ in the top and bottom,

$$
I_{2 n}=\frac{(2 n-1)(2 n-2)(2 n-3)(2 n-4)(2 n-5) \cdots(6)(5)(4)(3)(2)(1)}{(2 n)((2 n-2)(2 n-4) \cdots(6)(4)(2))^{2}} \frac{\pi}{2}=\frac{(2 n-1)!}{2 n\left(2^{n-1}(n-1)!\right)^{2}} \frac{\pi}{2} .
$$

Applying De Moivre's asymptotic formula $n!\sim C(n / e)^{n} \sqrt{n}$,

$$
I_{2 n} \sim \frac{C((2 n-1) / e)^{2 n-1} \sqrt{2 n-1}}{2 n\left(2^{n-1} C((n-1) / e)^{n-1} \sqrt{n-1}\right)^{2}} \frac{\pi}{2}=\frac{(2 n-1)^{2 n} \frac{1}{2 n-1} \sqrt{2 n-1}}{2 n \cdot 2^{2(n-1)} C e(n-1)^{2 n} \frac{1}{(n-1)^{2}}(n-1)} \frac{\pi}{2}
$$

as $n \rightarrow \infty$. For any $a \in \mathbf{R},(1+a / n)^{n} \rightarrow e^{a}$ as $n \rightarrow \infty$, so $(n+a)^{n} \sim e^{a} n^{n}$. Substituting this into the above formula with $a=-1$ and $n$ replaced by $2 n$,

$$
\begin{equation*}
I_{2 n} \sim \frac{e^{-1}(2 n)^{2 n} \frac{1}{\sqrt{2 n}}}{2 n \cdot 2^{2(n-1)} C e\left(e^{-1} n^{n}\right)^{2} \frac{1}{n^{2}} n} \frac{\pi}{2}=\frac{\pi}{C \sqrt{2 n}} \tag{8.1}
\end{equation*}
$$

Since $I_{k-1} \sim I_{k}$, the outer terms in (7.3) are both asymptotic to $\sqrt{n} I_{2 n} \sim \pi /(C \sqrt{2})$ by (8.1). Therefore

$$
\int_{0}^{\sqrt{n}} e^{-y^{2}} \mathrm{~d} y \rightarrow \frac{\pi}{C \sqrt{2}}
$$

as $n \rightarrow \infty$, so $J=\pi /(C \sqrt{2})$. Therefore $C=\sqrt{2 \pi}$ if and only if $J=\sqrt{\pi} / 2$.

## 9. Ninth Proof: The original proof

The original proof that $J=\sqrt{\pi} / 2$ is due to Laplace [8] in 1774. (An English translation of Laplace's article is mentioned in the bibliographic citation for [8], with preliminary comments on that article in [18].) He wanted to compute

$$
\begin{equation*}
\int_{0}^{1} \frac{\mathrm{~d} x}{\sqrt{-\log x}} \tag{9.1}
\end{equation*}
$$

Setting $y=\sqrt{-\log x}$, this integral is $2 \int_{0}^{\infty} e^{-y^{2}} \mathrm{~d} y=2 J$, so we expect (9.1) to be $\sqrt{\pi}$.
Laplace's starting point for evaluating (9.1) was a formula of Euler:

$$
\begin{equation*}
\int_{0}^{1} \frac{x^{r} \mathrm{~d} x}{\sqrt{1-x^{2 s}}} \int_{0}^{1} \frac{x^{s+r} \mathrm{~d} x}{\sqrt{1-x^{2 s}}}=\frac{1}{s(r+1)} \frac{\pi}{2} \tag{9.2}
\end{equation*}
$$

for positive $r$ and $s$. (Laplace himself said this formula held "whatever be" $r$ or $s$, but if $s<0$ then the number under the square root is negative.) Accepting (9.2), let $r \rightarrow 0$ in it to get

$$
\begin{equation*}
\int_{0}^{1} \frac{\mathrm{~d} x}{\sqrt{1-x^{2 s}}} \int_{0}^{1} \frac{x^{s} \mathrm{~d} x}{\sqrt{1-x^{2 s}}}=\frac{1}{s} \frac{\pi}{2} . \tag{9.3}
\end{equation*}
$$

Now let $s \rightarrow 0$ in (9.3). Then $1-x^{2 s} \sim-2 s \log x$ by L'Hopital's rule, so (9.3) becomes

$$
\left(\int_{0}^{1} \frac{\mathrm{~d} x}{\sqrt{-\log x}}\right)^{2}=\pi
$$

Thus (9.1) is $\sqrt{\pi}$.
Euler's formula (9.2) looks mysterious, but we have met it before. In the formula let $x^{s}=\cos \theta$ with $0 \leq \theta \leq \pi / 2$. Then $x=(\cos \theta)^{1 / s}$, and after some calculations (9.2) turns into

$$
\begin{equation*}
\int_{0}^{\pi / 2}(\cos \theta)^{(r+1) / s-1} \mathrm{~d} \theta \int_{0}^{\pi / 2}(\cos \theta)^{(r+1) / s} \mathrm{~d} \theta=\frac{1}{(r+1) / s} \frac{\pi}{2} \tag{9.4}
\end{equation*}
$$

We used the integral $I_{k}=\int_{0}^{\pi / 2}(\cos \theta)^{k} \mathrm{~d} \theta$ before when $k$ is a nonnegative integer. This notation makes sense when $k$ is any positive real number, and then (9.4) assumes the form $I_{\alpha} I_{\alpha+1}=\frac{1}{\alpha+1} \frac{\pi}{2}$ for $\alpha=(r+1) / s-1$, which is (7.5) with a possibly nonintegral index. Letting $r=0$ and $s=1 /(2 n+1)$ in (9.4) recovers (7.5). Letting $s \rightarrow 0$ in (9.3) corresponds to letting $n \rightarrow \infty$ in (7.5), so the proof in Section 7 is in essence a more detailed version of Laplace's 1774 argument.

## 10. Tenth Proof: Residue theorem

We will calculate $\int_{-\infty}^{\infty} e^{-x^{2} / 2} \mathrm{~d} x$ using contour integrals and the residue theorem. However, we can't just integrate $e^{-z^{2} / 2}$, as this function has no poles. For a long time nobody knew how to handle this integral using contour integration. For instance, in 1914 Watson [20, p. 79] wrote "Cauchy's theorem cannot be employed to evaluate all definite integrals; thus $\int_{0}^{\infty} e^{-x^{2}} \mathrm{~d} x$ has not been evaluated except by other methods." In the 1940s several contour integral solutions were published using awkward contours such as parallelograms [10], [12, Sect. 5] (see [2, Exer. 9, p. 113] for a recent appearance). Our approach will follow Kneser [6, p. 121] (see also [13, pp. 413-414] or [22]), using a rectangular contour and the function

$$
\frac{e^{-z^{2} / 2}}{1-e^{-\sqrt{\pi}(1+i) z}} .
$$

This function comes out of nowhere, so our first task is to motivate the introduction of this function.
We seek a meromorphic function $f(z)$ to integrate around the rectangular contour $\gamma_{R}$ in the figure below, with vertices at $-R, R, R+i b$, and $-R+i b$, where $b$ will be fixed and we let $R \rightarrow \infty$.


Suppose $f(z) \rightarrow 0$ along the right and left sides of $\gamma_{R}$ uniformly as $R \rightarrow \infty$. Then by applying the residue theorem and letting $R \rightarrow \infty$, we would obtain (if the integrals converge)

$$
\int_{-\infty}^{\infty} f(x) \mathrm{d} x+\int_{\infty}^{-\infty} f(x+i b) \mathrm{d} x=2 \pi i \sum_{a} \operatorname{Res}_{z=a} f(z)
$$

where the sum is over poles of $f(z)$ with imaginary part between 0 and $b$. This is equivalent to

$$
\int_{-\infty}^{\infty}(f(x)-f(x+i b)) \mathrm{d} x=2 \pi i \sum_{a} \operatorname{Res}_{z=a} f(z)
$$

Therefore we want $f(z)$ to satisfy

$$
\begin{equation*}
f(z)-f(z+i b)=e^{-z^{2} / 2} \tag{10.1}
\end{equation*}
$$

where $f(z)$ and $b$ need to be determined.
Let's try $f(z)=e^{-z^{2} / 2} / d(z)$, for an unknown denominator $d(z)$ whose zeros are poles of $f(z)$. We want $f(z)$ to satisfy

$$
\begin{equation*}
f(z)-f(z+\tau)=e^{-z^{2} / 2} \tag{10.2}
\end{equation*}
$$

for some $\tau$ (which will not be purely imaginary, so (10.1) doesn't quite work, but (10.1) is only motivation). Substituting $e^{-z^{2} / 2} / d(z)$ for $f(z)$ in (10.2) gives us

$$
\begin{equation*}
e^{-z^{2} / 2}\left(\frac{1}{d(z)}-\frac{e^{-\tau z-\tau^{2} / 2}}{d(z+\tau)}\right)=e^{-z^{2} / 2} \tag{10.3}
\end{equation*}
$$

Suppose $d(z+\tau)=d(z)$. Then (10.3) implies

$$
d(z)=1-e^{-\tau z-\tau^{2} / 2},
$$

and with this definition of $d(z), e^{-z^{2} / 2} / d(z)$ satisfies (10.2) if and only if $e^{\tau^{2}}=1$, or equivalently $\tau^{2} \in 2 \pi i \mathbf{Z}$. The simplest nonzero solution is $\tau=\sqrt{\pi}(1+i)$. From now on this is the value of $\tau$, so $e^{-\tau^{2} / 2}=e^{-i \pi}=-1$ and $d(z)=1+e^{-\tau z}$. Set

$$
f(z)=\frac{e^{-z^{2} / 2}}{d(z)}=\frac{e^{-z^{2} / 2}}{1+e^{-\tau z}},
$$

which is Kneser's function mentioned earlier. This function satisfies (10.2) and we henceforth ignore the motivation (10.1). Poles of $f(z)$ are at odd integral multiples of $\tau / 2$.

We will integrate this $f(z)$ around the rectangular contour $\gamma_{R}$ below, whose height is $\operatorname{Im}(\tau)$.


The poles of $f(z)$ nearest the origin are plotted in the figure; they lie along the line $y=x$. The only pole of $f(z)$ inside $\gamma_{R}$ (for $R>\sqrt{\pi} / 2$ ) is at $\tau / 2$, so by the residue theorem

$$
\int_{\gamma_{R}} f(z) \mathrm{d} z=2 \pi i \operatorname{Res}_{z=\tau / 2} f(z)=2 \pi i \frac{e^{-\tau^{2} / 8}}{(-\tau) e^{-\tau^{2} / 2}}=\frac{2 \pi i e^{3 \tau^{2} / 8}}{-\sqrt{\pi}(1+i)}=\frac{2 \pi i e^{3 \pi i / 4}}{-\sqrt{\pi}(1+i)}=\sqrt{2 \pi}
$$

Since the left and right sides of $\gamma_{R}$ have the same length, $\sqrt{\pi}$, for all $R$, to show the integral of $f$ along those sides tends to 0 uniformly as $R \rightarrow \infty$, it suffices to show $f(z) \rightarrow 0$ uniformly along those sides as $R \rightarrow \infty$. Parametrize $z$ along the left and right sides as $-R+i t$ and $R+i t$ with $t$ running over $[0, \sqrt{\pi}]$ in one direction or the other (which won't matter since we'll be taking absolute values). Then, using the reverse triangle inequality in the denominator, when $R>\sqrt{\pi}$ (so $R>t$ )

$$
|f(R+i t)|=\frac{\left|e^{-R^{2} / 2-i R t+t^{2} / 2}\right|}{\left|1+e^{-\tau(R+i t)}\right|} \leq \frac{e^{-R^{2} / 2} e^{t^{2} / 2}}{\left|1-e^{-\operatorname{Re}(\tau(R+i t))}\right|} \leq \frac{e^{-R^{2} / 2} e^{\pi / 2}}{1-e^{-\sqrt{\pi}(R-t)}}<\frac{e^{-R^{2} / 2} e^{\pi / 2}}{1-e^{-\sqrt{\pi}(R-\sqrt{\pi})}},
$$

which tends to 0 as $R \rightarrow \infty$. Also

$$
|f(-R+i t)|=\frac{\left|e^{-R^{2} / 2+i R t+t^{2} / 2}\right|}{\left|1+e^{-\tau(-R+i t)}\right|} \leq \frac{e^{-R^{2} / 2} e^{t^{2} / 2}}{\left|1-e^{-\operatorname{Re}(\tau(-R+i t))}\right|} \leq \frac{e^{-R^{2} / 2} e^{\pi / 2}}{e^{\sqrt{\pi}(R+t)}-1}<\frac{e^{-R^{2} / 2} e^{\pi / 2}}{e^{\sqrt{\pi} R}-1}
$$

which tends to 0 as $R \rightarrow \infty$. Thus

$$
\sqrt{2 \pi}=\int_{-\infty}^{\infty} f(x) \mathrm{d} x+\int_{\infty+i \sqrt{\pi}}^{-\infty+i \sqrt{\pi}} f(z) \mathrm{d} z=\int_{-\infty}^{\infty} f(x) \mathrm{d} x-\int_{-\infty}^{\infty} f(x+i \sqrt{\pi}) \mathrm{d} x .
$$

In the second integral, write $i \sqrt{\pi}$ as $\tau-\pi$ and use (real) translation invariance of $\mathrm{d} x$ to obtain

$$
\sqrt{2 \pi}=\int_{-\infty}^{\infty} f(x) \mathrm{d} x-\int_{-\infty}^{\infty} f(x+\tau) \mathrm{d} x=\int_{-\infty}^{\infty}(f(x)-f(x+\tau)) \mathrm{d} x=\int_{-\infty}^{\infty} e^{-x^{2} / 2} \mathrm{~d} x \quad \text { by (10.2). }
$$

## 11. Eleventh Proof: Fourier transforms

For a continuous function $f: \mathbf{R} \rightarrow \mathbf{C}$ that is rapidly decreasing at $\pm \infty$, its Fourier transform is the function $\mathcal{F} f: \mathbf{R} \rightarrow \mathbf{C}$ defined by

$$
\begin{equation*}
(\mathcal{F} f)(y)=\int_{-\infty}^{\infty} f(x) e^{-i x y} \mathrm{~d} x . \tag{11.1}
\end{equation*}
$$

For example, $(\mathcal{F} f)(0)=\int_{-\infty}^{\infty} f(x) \mathrm{d} x$.
Here are three properties of the Fourier transform.

- If $f$ is differentiable, then after using differentiation under the integral sign on the Fourier transform of $f$ we obtain

$$
(\mathcal{F} f)^{\prime}(y)=\int_{-\infty}^{\infty}-i x f(x) e^{-i x y} \mathrm{~d} x=-i(\mathcal{F}(x f(x)))(y)
$$

- Using integration by parts on the Fourier transform of $f$, with $u=f(x)$ and $\mathrm{d} v=e^{-i x y} \mathrm{~d} x$, we obtain

$$
\left(\mathcal{F}\left(f^{\prime}\right)\right)(y)=i y(\mathcal{F} f)(y) .
$$

- If we apply the Fourier transform twice then we recover the original function up to interior and exterior scaling:

$$
\begin{equation*}
\left(\mathcal{F}^{2} f\right)(x)=2 \pi f(-x) . \tag{11.2}
\end{equation*}
$$

The $2 \pi$ is admittedly a nonobvious scaling factor here, and the proof of (11.2) is nontrivial. We'll show the appearance of $2 \pi$ in (11.2) is equivalent to the evaluation of $I$ as $\sqrt{2 \pi}$.

Fixing $a>0$, set $f(x)=e^{-a x^{2}}$, so

$$
f^{\prime}(x)=-2 a x f(x) .
$$

Applying the Fourier transform to both sides of this equation implies $i y(\mathcal{F} f)(y)=-2 a \frac{1}{-i}(\mathcal{F} f)^{\prime}(y)$, which simplifies to $(\mathcal{F} f)^{\prime}(y)=-\frac{1}{2 a} y(\mathcal{F} f)(y)$. The general solution of $g^{\prime}(y)=-\frac{1}{2 a} y g(y)$ is $g(y)=$ $C e^{-y^{2} /(4 a)}$, so

$$
f(x)=e^{-a x^{2}} \Longrightarrow(\mathcal{F} f)(y)=C e^{-y^{2} /(4 a)}
$$

for some constant $C$. We have $1 /(4 a)=a$ when $a=1 / 2$, so set $a=1 / 2$ : if $f(x)=e^{-x^{2} / 2}$ then

$$
\begin{equation*}
(\mathcal{F} f)(y)=C e^{-y^{2} / 2}=C f(y) . \tag{11.3}
\end{equation*}
$$

Setting $y=0$ in (11.3), the left side is $(\mathcal{F} f)(0)=\int_{-\infty}^{\infty} e^{-x^{2} / 2} \mathrm{~d} x=I$, so $I=C f(0)=C$.
Applying the Fourier transform to both sides of (11.3) with $C=I$ and using (11.2), we get $2 \pi f(-x)=I(\mathcal{F} f)(x)=I^{2} f(x)$. At $x=0$ this becomes $2 \pi=I^{2}$, so $I=\sqrt{2 \pi}$ since $I>0$. That is the Gaussian integral calculation. If we didn't know that the constant on the right side of (11.2) is $2 \pi$, whatever its value is would wind up being $I^{2}$, so saying $2 \pi$ appears on the right side of (11.2) is equivalent to saying $I=\sqrt{2 \pi}$.

There are other ways to define the Fourier transform besides (11.1), such as

$$
\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(x) e^{-i x y} \mathrm{~d} x \quad \text { or } \quad \int_{-\infty}^{\infty} f(x) e^{-2 \pi i x y} \mathrm{~d} x
$$

These transforms have properties similar to the transform as defined in (11.1), so they can be used in its place to compute the Gaussian integral. Let's see how such a proof looks using the second alternative definition, which we'll write as

$$
(\widetilde{\mathcal{F}} f)(y)=\int_{-\infty}^{\infty} f(x) e^{-2 \pi i x y} \mathrm{~d} x .
$$

For this Fourier transform, the analogue of the three properties above for $\mathcal{F}$ are

- $(\widetilde{\mathcal{F}} f)^{\prime}(y)=-2 \pi i(\widetilde{\mathcal{F}}(x f(x)))(y)$.
- $\left(\widetilde{\mathcal{F}}\left(f^{\prime}\right)\right)(y)=2 \pi i y(\widetilde{\mathcal{F}} f)(y)$.
- $\left(\widetilde{\mathcal{F}}^{2} f\right)(x)=f(-x)$.

The last property for $\widetilde{\mathcal{F}}$ looks nicer than that for $\mathcal{F}$, since there is no overall $2 \pi$-factor on the right side (it has been hidden in the definition of $\widetilde{\mathcal{F}}$ ). On the other hand, the first two properties for $\widetilde{\mathcal{F}}$ have overall factors of $2 \pi$ on the right side while the first two properties of $\mathcal{F}$ do not. You can't escape a role for $\pi$ or $2 \pi$ somewhere in every possible definition of a Fourier transform.

Now let's run through the proof again with $\widetilde{\mathcal{F}}$ in place of $\mathcal{F}$. For $a>0$, set $f(x)=e^{-a x^{2}}$. Applying $\widetilde{\mathcal{F}}$ to both sides of the equation $f^{\prime}(x)=-2 a x f(x), 2 \pi i y(\widetilde{\mathcal{F}} f)(y)=-2 a \frac{1}{-(2 \pi i)}(\mathcal{F} f)^{\prime}(y)$, and that is equivalent to $(\widetilde{\mathcal{F}} f)^{\prime}(y)=-\frac{2 \pi^{2}}{a} y(\mathcal{F} f)(y)$. Solutions of $g^{\prime}(y)=-\frac{2 \pi^{2}}{a} y g(y)$ all look like $C e^{-\left(\pi^{2} / a\right) y^{2}}$, so

$$
f(x)=e^{-a x^{2}} \Longrightarrow(\widetilde{\mathcal{F}} f)(y)=C e^{-\left(\pi^{2} / a\right) y^{2}}
$$

for a constant $C$. We want $\pi^{2} / a=\pi$ so that $e^{-\left(\pi^{2} / a\right) y^{2}}=e^{-\pi y^{2}}=f(y)$, which occurs for $a=\pi$. Thus when $f(x)=e^{-\pi x^{2}}$ we have

$$
\begin{equation*}
(\widetilde{\mathcal{F}} f)(y)=C e^{-\pi y^{2}}=C f(y) . \tag{11.4}
\end{equation*}
$$

When $y=0$ in (11.4), this becomes $\int_{-\infty}^{\infty} e^{-\pi x^{2}} \mathrm{~d} x=C$, so $C=K$ : see the top of the first page for the definition of $K$ as the integral of $e^{-\pi x^{2}}$ over $\mathbf{R}$.

Applying $\widetilde{\mathcal{F}}$ to both sides of (11.4) with $C=K$ and using $\left(\widetilde{\mathcal{F}}^{2} f\right)(x)=f(-x)$, we get $f(-x)=$ $K(\widetilde{\mathcal{F}} f)(x)=K^{2} f(x)$. At $x=0$ this becomes $1=K^{2}$, so $K=1$ since $K>0$. That $K=1$, or in more explicit form $\int_{-\infty}^{\infty} e^{-\pi x^{2}} \mathrm{~d} x=1$, is equivalent to the evaluation of the Gaussian integral $I$ with the change of variables $y=\sqrt{2 \pi} x$ in the integral for $K$.

## 12. History of the Gaussian integral

The function $e^{-x^{2} / 2}$, or in the form $\frac{1}{\sqrt{2 \pi}} e^{-x^{2} / 2}$ ("normal distribution") to have total integral 1 over $\mathbf{R}$, plays an essential role in probability and statistics, and it was in probabilistic settings that it was first found. The approximation of a binomial distribution with many samples by a normal distribution, which is a mainstay of probability courses today, is how the normal distribution was first found in work of De Moivre in 1733. This role as a mere approximation did not make it stand out. In the 1770 s

## Appendix A. Redoing Section 2 without improper integrals in Fubini's theorem

In this appendix we will work out the calculation of the Gaussian integral in Section 2 without relying on Fubini's theorem for improper integrals. The key equation is (2.1), which we recall:

$$
\int_{0}^{\infty}\left(\int_{0}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y=\int_{0}^{\infty}\left(\int_{0}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} y\right) \mathrm{d} t
$$

The calculation in Section 2 that the iterated integral on the right is $\pi / 4$ does not need Fubini's theorem in any form. It is going from the iterated integral on the left to $\pi / 4$ that used Fubini's theorem for improper integrals. The next theorem could be used as a substitute, and its proof will only use Fubini's theorem for integrals on rectangles.

Theorem A.1. For $b>1$ and $c>1$,

$$
\int_{0}^{\infty}\left(\int_{0}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y=\frac{\pi}{4}+O\left(\frac{1}{b}\right)+O\left(\frac{1}{\sqrt{c}}\right) .
$$

Having $b \rightarrow \infty$ and $c \rightarrow \infty$ in Theorem A. 1 makes the right side $\pi / 4$ without changing the left side.

Lemma A.2. (1) For all $x \in \mathbf{R}, e^{-x^{2}} \leq \frac{1}{x^{2}+1}$.
(2) For $a>0 \int_{0}^{\infty} \frac{\mathrm{d} x}{a^{2} x^{2}+1}=\frac{\pi}{2 a}$.
(3) For $a>0$ and $c>0, \int_{c}^{\infty} \frac{\mathrm{d} x}{a^{2} x^{2}+1}=\frac{1}{a}\left(\frac{\pi}{2}-\arctan (a c)\right)$.
(4) For $a>0$ and $c>0, \int_{c}^{\infty} \frac{\mathrm{d} x}{a^{2} x^{2}+1}<\frac{1}{a^{2} c}$.
(5) For $a>0, \frac{\pi}{2}-\arctan a<\frac{1}{a}$.

Proof. The proofs of (1), (2), and (3) are left to the reader. To prove (4), replace $1+a^{2} t^{2}$ by the smaller value $a^{2} t^{2}$. To prove (5), write the difference as $\int_{a}^{\infty} d x /\left(x^{2}+1\right)$ and then bound $1 /\left(x^{2}+1\right)$ above by $1 / x^{2}$.

Now we prove Theorem A.1.
Proof. Step 1. For $b>1$ and $c>1$, we'll show the improper integral can be truncated to an integral over $[0, \overline{b] \times[0}, c]$ plus error terms:

$$
\int_{0}^{\infty}\left(\int_{0}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y=\int_{0}^{b}\left(\int_{0}^{c} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y+O\left(\frac{1}{\sqrt{c}}\right)+O\left(\frac{1}{b}\right) .
$$

Subtract the integral on the right from the integral on the left and split the outer integral $\int_{0}^{\infty}$ into $\int_{0}^{b}+\int_{b}^{\infty}$ :

$$
\begin{aligned}
\int_{0}^{\infty}\left(\int_{0}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y-\int_{0}^{b}\left(\int_{0}^{c} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y & =\int_{0}^{b}\left(\int_{c}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y \\
& +\int_{b}^{\infty}\left(\int_{0}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y
\end{aligned}
$$

On the right side, we will show the first iterated integral is $O(1 / \sqrt{c})$ and the second iterated integral is $O(1 / b)$. The second iterated integral is simpler:

$$
\begin{aligned}
\int_{b}^{\infty}\left(\int_{0}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y & =\int_{b}^{\infty}\left(\int_{0}^{\infty} e^{-(y t)^{2}} \mathrm{~d} t\right) y e^{-y^{2}} \mathrm{~d} y \\
& \leq \int_{b}^{\infty}\left(\int_{0}^{\infty} \frac{\mathrm{d} t}{y^{2} t^{2}+1}\right) y e^{-y^{2}} \mathrm{~d} y \quad \text { by Lemma A.2(1) } \\
& =\int_{b}^{\infty} \frac{\pi}{2 y} y e^{-y^{2}} \mathrm{~d} y \quad \text { by Lemma A.2(2) } \\
& =\frac{\pi}{2} \int_{b}^{\infty} e^{-y^{2}} \mathrm{~d} y \\
& \leq \frac{\pi}{2} \int_{b}^{\infty} \frac{\mathrm{d} y}{y^{2}+1} \quad \text { by Lemma A.2(1) } \\
& =\frac{\pi}{2 b} \quad \text { since } \frac{1}{y^{2}+1}<\frac{1}{y^{2}},
\end{aligned}
$$

and this is $O(1 / b)$. Returning to the first iterated integral,

$$
\begin{aligned}
\int_{0}^{b}\left(\int_{c}^{\infty} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y & =\int_{0}^{b}\left(\int_{c}^{\infty} e^{-(y t)^{2}} \mathrm{~d} t\right) y e^{-y^{2}} \mathrm{~d} y \\
& \leq \int_{0}^{b}\left(\int_{c}^{\infty} \frac{\mathrm{d} t}{y^{2} t^{2}+1}\right) y e^{-y^{2}} \mathrm{~d} y \quad \text { by Lemma A.2(1) } \\
& =\int_{0}^{1}\left(\int_{c}^{\infty} \frac{\mathrm{d} t}{y^{2} t^{2}+1}\right) y e^{-y^{2}} \mathrm{~d} y+\int_{1}^{b}\left(\int_{c}^{\infty} \frac{\mathrm{d} t}{y^{2} t^{2}+1}\right) y e^{-y^{2}} \mathrm{~d} y \\
& \leq \int_{0}^{1}\left(\int_{c}^{\infty} \frac{\mathrm{d} t}{y^{2} t^{2}+1}\right) y e^{-y^{2}} \mathrm{~d} y+\int_{1}^{b} \frac{1}{y^{2} c} y e^{-y^{2}} \mathrm{~d} y \quad \text { by Lemma A.2(4) } \\
& =\int_{0}^{1}\left(\frac{\pi}{2}-\arctan (y c)\right) e^{-y^{2}} \mathrm{~d} y+\frac{1}{c} \int_{1}^{b} \frac{\mathrm{~d} y}{y e^{y^{2}}} \quad \text { by Lemma A.2(3) } \\
& \leq \int_{0}^{1}\left(\frac{\pi}{2}-\arctan (y c)\right) \mathrm{d} y+\frac{1}{c} \int_{1}^{\infty} \frac{\mathrm{d} y}{y e^{y^{2}}}
\end{aligned}
$$

The last term is $O(1 / c)$. We will show the first term is $O(1 / \sqrt{c})$ by carefully splitting up $\int_{0}^{1}$.
For $0<\varepsilon<1$,

$$
\int_{0}^{1}\left(\frac{\pi}{2}-\arctan (y c)\right) \mathrm{d} y=\int_{0}^{\varepsilon}\left(\frac{\pi}{2}-\arctan (y c)\right) \mathrm{d} y+\int_{\varepsilon}^{1}\left(\frac{\pi}{2}-\arctan (y c)\right) \mathrm{d} y .
$$

Both integrals are positive, and the first one is less than $(\pi / 2) \varepsilon$. The integrand of the second integral is less than $1 /(y c)$ by Lemma A.2(5), so

$$
\int_{\varepsilon}^{1}\left(\frac{\pi}{2}-\arctan (y c)\right) \mathrm{d} y<\int_{\varepsilon}^{1} \frac{\mathrm{~d} y}{y c}<\frac{1-\varepsilon}{\varepsilon c}<\frac{1}{\varepsilon c} .
$$

Therefore

$$
0<\int_{0}^{1}\left(\frac{\pi}{2}-\arctan (y c)\right) \mathrm{d} y<\frac{\pi}{2} \varepsilon+\frac{1}{\varepsilon c}
$$

for each $\varepsilon$ in $(0,1)$. Use $\varepsilon=1 / \sqrt{c}$ to get

$$
0<\int_{0}^{1}\left(\frac{\pi}{2}-\arctan (y c)\right) \mathrm{d} y<\frac{\pi}{2 \sqrt{c}}+\frac{1}{\sqrt{c}}=O\left(\frac{1}{\sqrt{c}}\right) .
$$

That proves the first iterated integral is $O(1 / \sqrt{c})+O(1 / c)=O(1 / \sqrt{c})$ as $c \rightarrow \infty$.
Step 2. For $b>0$ and $c>0$, we will show

$$
\int_{0}^{b}\left(\int_{0}^{c} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y=\frac{\pi}{4}+O\left(\frac{1}{e^{b^{2}}}\right)+O\left(\frac{1}{c}\right) .
$$

By Fubini's theorem for continuous functions on a rectangle in $\mathbf{R}^{2}$,

$$
\int_{0}^{b}\left(\int_{0}^{c} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} t\right) \mathrm{d} y=\int_{0}^{c}\left(\int_{0}^{b} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} y\right) \mathrm{d} t
$$

For the inner integral on the right, the formula $\int_{0}^{b} y e^{-a y^{2}} \mathrm{~d} y=1 /(2 a)-1 /\left(2 a e^{a b^{2}}\right)$ for $a>0$ tells us

$$
\int_{0}^{b} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} y=\frac{1}{2\left(t^{2}+1\right)}-\frac{1}{2\left(t^{2}+1\right) e^{\left(t^{2}+1\right) b^{2}}}
$$

SO

$$
\begin{align*}
\int_{0}^{c}\left(\int_{0}^{b} y e^{-\left(t^{2}+1\right) y^{2}} \mathrm{~d} y\right) \mathrm{d} t & =\frac{1}{2} \int_{0}^{c} \frac{\mathrm{~d} t}{t^{2}+1}-\frac{1}{2} \int_{0}^{c} \frac{\mathrm{~d} t}{\left(t^{2}+1\right) e^{\left(t^{2}+1\right) b^{2}}} \\
& =\frac{1}{2} \arctan (c)-\frac{1}{2} \int_{0}^{c} \frac{\mathrm{~d} t}{\left(t^{2}+1\right) e^{\left(t^{2}+1\right) b^{2}}} \tag{A.1}
\end{align*}
$$

Let's estimate these last two terms. Since

$$
\arctan (c)=\int_{0}^{\infty} \frac{\mathrm{d} t}{t^{2}+1}-\int_{c}^{\infty} \frac{\mathrm{d} t}{t^{2}+1}=\frac{\pi}{2}+O\left(\int_{c}^{\infty} \frac{\mathrm{d} t}{t^{2}}\right)=\frac{\pi}{2}+O\left(\frac{1}{c}\right)
$$

and

$$
\int_{0}^{c} \frac{\mathrm{~d} t}{\left(t^{2}+1\right) e^{\left(t^{2}+1\right) b^{2}}} \leq \int_{0}^{c} \frac{\mathrm{~d} t}{t^{2}+1} \frac{1}{e^{b^{2}}} \leq \int_{0}^{\infty} \frac{\mathrm{d} t}{t^{2}+1} \frac{1}{e^{b^{2}}}=O\left(\frac{1}{e^{b^{2}}}\right)
$$

feeding these error estimates into (A.1) finishes Step 2.

## References

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[^0]:    ${ }^{1}$ For a visualization of this calculation as a volume, in terms of $\int_{-\infty}^{\infty} e^{-x^{2}} \mathrm{~d} x$ instead of $J$, see https://www. youtube.com/watch?v=cy8r7WSuT1I. We'll do a volume calculation for $I^{2}$ in Section 5.

