## THE GAUSSIAN INTEGRAL

#### KEITH CONRAD

Let

$$I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, \mathrm{d}x, \ J = \int_{0}^{\infty} e^{-x^2} \, \mathrm{d}x, \ \text{and} \ K = \int_{-\infty}^{\infty} e^{-\pi x^2} \, \mathrm{d}x$$

These positive numbers are related:  $J = I/(2\sqrt{2})$  and  $K = I/\sqrt{2\pi}$ .

**Theorem.** With notation as above,  $I = \sqrt{2\pi}$ , or equivalently  $J = \sqrt{\pi}/2$ , or equivalently K = 1.

We will give multiple proofs of this. (Other lists of proofs are in [4] and [9].) It is subtle since  $e^{-\frac{1}{2}x^2}$  has no simple antiderivative. For comparison,  $\int_0^\infty x e^{-\frac{1}{2}x^2} dx$  can be computed with the antiderivative  $-e^{-\frac{1}{2}x^2}$  and equals 1. In the last section, the Gaussian integral's history is presented.

# 1. FIRST PROOF: POLAR COORDINATES

The most widely known proof, due to Poisson [9, p. 3], expresses  $J^2$  as a double integral and then uses polar coordinates. To start, write  $J^2$  as an iterated integral using single-variable calculus:

$$J^{2} = J \int_{0}^{\infty} e^{-y^{2}} dy = \int_{0}^{\infty} J e^{-y^{2}} dy = \int_{0}^{\infty} \left( \int_{0}^{\infty} e^{-x^{2}} dx \right) e^{-y^{2}} dy = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x^{2}+y^{2})} dx dy.$$

View this as a double integral over the first quadrant. To compute it with polar coordinates, the first quadrant is  $\{(r, \theta) : r \ge 0 \text{ and } 0 \le \theta \le \pi/2\}$ . Writing  $x^2 + y^2$  as  $r^2$  and dx dy as  $r dr d\theta$ ,

$$J^{2} = \int_{0}^{\pi/2} \int_{0}^{\infty} e^{-r^{2}} r \, \mathrm{d}r \, \mathrm{d}\theta$$
$$= \int_{0}^{\infty} r e^{-r^{2}} \, \mathrm{d}r \cdot \int_{0}^{\pi/2} \, \mathrm{d}\theta$$
$$= -\frac{1}{2} e^{-r^{2}} \Big|_{0}^{\infty} \cdot \frac{\pi}{2}$$
$$= \frac{1}{2} \cdot \frac{\pi}{2}$$
$$= \frac{\pi}{4}.$$

Since J > 0,  $J = \sqrt{\pi}/2$ .<sup>1</sup> It is argued in [1] that this method can't be used on any other integral.

2. Second Proof: Another change of variables

Our next proof uses another change of variables to compute  $J^2$ . As before,

$$J^2 = \int_0^\infty \left( \int_0^\infty e^{-(x^2 + y^2)} \,\mathrm{d}x \right) \,\mathrm{d}y.$$

<sup>&</sup>lt;sup>1</sup>For a visualization of this calculation as a volume, in terms of  $\int_{-\infty}^{\infty} e^{-x^2} dx$  instead of *J*, see https://www.youtube.com/watch?v=cy8r7WSuT1I. We'll do a volume calculation for  $I^2$  in Section 5.

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Instead of using polar coordinates, set x = yt in the inner integral (y is fixed). Then dx = y dt and

(2.1) 
$$J^{2} = \int_{0}^{\infty} \left( \int_{0}^{\infty} e^{-y^{2}(t^{2}+1)} y \, \mathrm{d}t \right) \, \mathrm{d}y = \int_{0}^{\infty} \left( \int_{0}^{\infty} y e^{-y^{2}(t^{2}+1)} \, \mathrm{d}y \right) \, \mathrm{d}t$$

where the interchange of integrals is justified by Fubini's theorem for improper Riemann integrals. (The appendix gives an approach using Fubini's theorem for Riemann integrals on rectangles.) Since  $\int_0^\infty y e^{-ay^2} dy = \frac{1}{2a}$  for a > 0, we have

$$J^{2} = \int_{0}^{\infty} \frac{\mathrm{d}t}{2(t^{2}+1)} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4},$$

so  $J = \sqrt{\pi}/2$ . This proof is due to Laplace [7, pp. 94–96] and historically precedes the widely used technique of the previous proof. We will see in Section 9 what Laplace's first proof was.

## 3. THIRD PROOF: DIFFERENTIATING UNDER THE INTEGRAL SIGN

For t > 0, set

$$A(t) = \left(\int_0^t e^{-x^2} \,\mathrm{d}x\right)^2.$$

The integral we want to calculate is  $A(\infty) = J^2$  and then take a square root.

Differentiating A(t) with respect to t and using the Fundamental Theorem of Calculus,

$$A'(t) = 2\int_0^t e^{-x^2} \,\mathrm{d}x \cdot e^{-t^2} = 2e^{-t^2}\int_0^t e^{-x^2} \,\mathrm{d}x.$$

Let x = ty, so

$$A'(t) = 2e^{-t^2} \int_0^1 te^{-t^2y^2} \, \mathrm{d}y = \int_0^1 2te^{-(1+y^2)t^2} \, \mathrm{d}y.$$

The function under the integral sign is easily antidifferentiated with respect to t:

$$A'(t) = \int_0^1 -\frac{\partial}{\partial t} \frac{e^{-(1+y^2)t^2}}{1+y^2} \, \mathrm{d}y = -\frac{d}{dt} \int_0^1 \frac{e^{-(1+y^2)t^2}}{1+y^2} \, \mathrm{d}y$$

Letting

$$B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2} \,\mathrm{d}x,$$

we have A'(t) = -B'(t) for all t > 0, so there is a constant C such that

(3.1) 
$$A(t) = -B(t) + C$$

for all t > 0. To find C, we let  $t \to 0^+$  in (3.1). The left side tends to  $\left(\int_0^0 e^{-x^2} dx\right)^2 = 0$  while the right side tends to  $-\int_0^1 dx/(1+x^2) + C = -\pi/4 + C$ . Thus  $C = \pi/4$ , so (3.1) becomes  $\left(\int_0^t e^{-x^2} dx\right)^2 = \frac{\pi}{2} - \int_0^1 \frac{e^{-t^2(1+x^2)}}{4t^2} dx$ 

$$\left(\int_0^{\infty} e^{-\frac{1}{4}} dx\right) = \frac{1}{4} - \int_0^{\infty} \frac{1}{1+x^2} dx$$

Letting  $t \to \infty$  in this equation, we obtain  $J^2 = \pi/4$ , so  $J = \sqrt{\pi/2}$ .

A comparison of this proof with the first proof is in [21].

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## 4. Fourth Proof: Another differentiation under the integral sign

Here is a second approach to finding J by differentiation under the integral sign. I heard about it from Michael Rozman [14], who modified an idea on math.stackexchange [23], and in a slightly less elegant form it appeared much earlier in [19].

For  $t \in \mathbf{R}$ , set

$$F(t) = \int_0^\infty \frac{e^{-t^2(1+x^2)}}{1+x^2} \,\mathrm{d}x.$$

Then  $F(0) = \int_0^\infty dx/(1+x^2) = \pi/2$  and  $F(\infty) = 0$ . Differentiating under the integral sign,

$$F'(t) = \int_0^\infty -2te^{-t^2(1+x^2)} \,\mathrm{d}x = -2te^{-t^2} \int_0^\infty e^{-(tx)^2} \,\mathrm{d}x$$

Make the substitution y = tx, with dy = t dx, so

$$F'(t) = -2e^{-t^2} \int_0^\infty e^{-y^2} \,\mathrm{d}y = -2Je^{-t^2}.$$

For b > 0, integrate both sides from 0 to b and use the Fundamental Theorem of Calculus:

$$\int_0^b F'(t) \, \mathrm{d}t = -2J \int_0^b e^{-t^2} \, \mathrm{d}t \Longrightarrow F(b) - F(0) = -2J \int_0^b e^{-t^2} \, \mathrm{d}t.$$

Letting  $b \to \infty$  in the last equation,

$$0 - \frac{\pi}{2} = -2J^2 \Longrightarrow J^2 = \frac{\pi}{4} \Longrightarrow J = \frac{\sqrt{\pi}}{2}.$$

## 5. FIFTH PROOF: A VOLUME INTEGRAL

Our next proof is due to T. P. Jameson [5] and it was rediscovered by A. L. Delgado [3]. Revolve the curve  $z = e^{-\frac{1}{2}x^2}$  in the *xz*-plane around the *z*-axis to produce the "bell surface"  $z = e^{-\frac{1}{2}(x^2+y^2)}$ . See below, where the *z*-axis is vertical and passes through the top point, the *x*-axis lies just under the surface through the point 0 in front, and the *y*-axis lies just under the surface through the point 0 on the left. We will compute the volume V below the surface and above the *xy*-plane in two ways.

First we compute V by *horizontal slices*, which are discs:  $V = \int_0^1 A(z) dz$  where A(z) is the area of the disc formed by slicing the surface at height z. Writing the radius of the disc at height z as r(z),  $A(z) = \pi r(z)^2$ . To compute r(z), the surface cuts the xz-plane at a pair of points  $(x, e^{-\frac{1}{2}x^2})$  where the height is z, so  $e^{-\frac{1}{2}x^2} = z$ . Thus  $x^2 = -2 \ln z$ . Since x is the distance of these points from the z-axis,  $r(z)^2 = x^2 = -2 \ln z$ , so  $A(z) = \pi r(z)^2 = -2\pi \ln z$ . Therefore

$$V = \int_0^1 -2\pi \ln z \, \mathrm{d}z = -2\pi \left(z \ln z - z\right) \Big|_0^1 = -2\pi (-1 - \lim_{z \to 0^+} z \ln z).$$

By L'Hospital's rule,  $\lim_{z\to 0^+} z \ln z = 0$ , so  $V = 2\pi$ . (A calculation of V by shells is in [11].)

Next we compute the volume by *vertical slices* in planes x = constant. Vertical slices are scaled bell curves: look at the black contour lines in the picture. The equation of the bell curve along the top of the vertical slice with x-coordinate x is  $z = e^{-\frac{1}{2}(x^2+y^2)}$ , where y varies and x is fixed. Then



 $V = \int_{-\infty}^{\infty} A(x) dx$ , where A(x) is the area of the x-slice:

$$A(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 + y^2)} \, \mathrm{d}y = e^{-\frac{1}{2}x^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \, \mathrm{d}y = e^{-\frac{1}{2}x^2} I.$$

Thus  $V = \int_{-\infty}^{\infty} A(x) dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} I dx = I \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = I^2.$ Comparing the two formulas for V, we have  $2\pi = I^2$ , so  $I = \sqrt{2\pi}$ .

6. Sixth Proof: The  $\Gamma$ -function

For any integer  $n \ge 0$ , we have  $n! = \int_0^\infty t^n e^{-t} dt$ . For x > 0 we define  $\Gamma(x) = \int_0^\infty t^x e^{-t} \frac{dt}{t},$ 

so  $\Gamma(n) = (n-1)!$  when  $n \ge 1$ . Using integration by parts,  $\Gamma(x+1) = x\Gamma(x)$ . One of the basic properties of the  $\Gamma$ -function [15, pp. 193–194] is

(6.1) 
$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Set x = y = 1/2:

$$\Gamma\left(\frac{1}{2}\right)^2 = \int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)}}$$

Note

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \sqrt{t}e^{-t}\frac{\mathrm{d}t}{t} = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t = \int_0^\infty \frac{e^{-x^2}}{x}2x\,\mathrm{d}x = 2\int_0^\infty e^{-x^2}\,\mathrm{d}x = 2J,$$

so  $4J^2 = \int_0^1 dt / \sqrt{t(1-t)}$ . With the substitution  $t = \sin^2 \theta$ ,

$$4J^2 = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\,\mathrm{d}\theta}{\sin\theta\cos\theta} = 2\frac{\pi}{2} = \pi,$$

so  $J = \sqrt{\pi}/2$ . Equivalently,  $\Gamma(1/2) = \sqrt{\pi}$ . Any method that proves  $\Gamma(1/2) = \sqrt{\pi}$  is also a method that calculates  $\int_0^\infty e^{-x^2} dx$ .

## 7. Seventh Proof: Asymptotic estimates

We will show  $J = \sqrt{\pi}/2$  by a technique whose steps are based on [16, p. 371].

For  $x \ge 0$ , power series expansions show  $1 + x \le e^x \le 1/(1 - x)$ . Reciprocating and replacing x with  $x^2$ , we get

(7.1) 
$$1 - x^2 \le e^{-x^2} \le \frac{1}{1 + x^2}.$$

for all  $x \in \mathbf{R}$ .

For any positive integer n, raise the terms in (7.1) to the nth power and integrate from 0 to 1:

$$\int_0^1 (1-x^2)^n \, \mathrm{d}x \le \int_0^1 e^{-nx^2} \, \mathrm{d}x \le \int_0^1 \frac{\mathrm{d}x}{(1+x^2)^n}$$

Using the changes of variables  $x = \sin \theta$  on the left,  $x = y/\sqrt{n}$  in the middle, and  $x = \tan \theta$  on the right,

(7.2) 
$$\int_0^{\pi/2} (\cos\theta)^{2n+1} \,\mathrm{d}\theta \le \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} \,\mathrm{d}y \le \int_0^{\pi/4} (\cos\theta)^{2n-2} \,\mathrm{d}\theta < \int_0^{\pi/2} (\cos\theta)^{2n-2} \,\mathrm{d}\theta.$$

Set  $I_k = \int_0^{\pi/2} (\cos \theta)^k d\theta$ , so  $I_0 = \pi/2$ ,  $I_1 = 1$ , and (7.2) implies

(7.3) 
$$\sqrt{n}I_{2n+1} \le \int_0^{\sqrt{n}} e^{-y^2} \,\mathrm{d}y < \sqrt{n}I_{2n-2}.$$

We will show that as  $k \to \infty$ ,  $kI_k^2 \to \pi/2$ . Then

$$\sqrt{n}I_{2n+1} = \frac{\sqrt{n}}{\sqrt{2n+1}}\sqrt{2n+1}I_{2n+1} \to \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2}$$

and

$$\sqrt{n}I_{2n-2} = \frac{\sqrt{n}}{\sqrt{2n-2}}\sqrt{2n-2}I_{2n-2} \to \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2},$$

so by (7.3),  $\int_0^{\sqrt{n}} e^{-y^2} dy \to \sqrt{\pi}/2$ . Thus  $J = \sqrt{\pi}/2$ .

To show  $kI_k^2 \rightarrow \pi/2$ , first we compute several values of  $I_k$  explicitly by a recursion. Using integration by parts,

$$I_{k} = \int_{0}^{\pi/2} (\cos \theta)^{k} \, \mathrm{d}\theta = \int_{0}^{\pi/2} (\cos \theta)^{k-1} \cos \theta \, \mathrm{d}\theta = (k-1)(I_{k-2} - I_{k}),$$

 $\mathbf{so}$ 

(7.4) 
$$I_k = \frac{k-1}{k} I_{k-2}$$

Using (7.4) and the initial values  $I_0 = \pi/2$  and  $I_1 = 1$ , the first few values of  $I_k$  are computed and listed in Table 1.

From Table 1 we see that

(7.5) 
$$I_{2n}I_{2n+1} = \frac{1}{2n+1}\frac{\pi}{2}$$

for  $0 \le n \le 3$ , and this can be proved for all *n* by induction using (7.4). Since  $0 \le \cos \theta \le 1$  for  $\theta \in [0, \pi/2]$ , we have  $I_k \le I_{k-1} \le I_{k-2} = \frac{k}{k-1}I_k$  by (7.4), so  $I_{k-1} \sim I_k$  as  $k \to \infty$ . Therefore (7.5) implies

$$I_{2n}^2 \sim \frac{1}{2n} \frac{\pi}{2} \Longrightarrow (2n) I_{2n}^2 \to \frac{\pi}{2}$$
$$(2n+1) I_{2n+1}^2 \sim (2n) I_{2n}^2 \to \frac{\pi}{2}$$

as  $n \to \infty$ . Then

as

$$n \to \infty$$
, so  $kI_k^2 \to \pi/2$  as  $k \to \infty$ . This completes our proof that  $J = \sqrt{\pi}/2$ .

**Remark 7.1.** This proof is closely related to the fifth proof using the  $\Gamma$ -function. Indeed, by (6.1)

$$\frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2}+\frac{1}{2})} = \int_0^1 t^{(k+1)/2+1} (1-t)^{1/2-1} \,\mathrm{d}t,$$

and with the change of variables  $t = (\cos \theta)^2$  for  $0 \le \theta \le \pi/2$ , the integral on the right is equal to  $2 \int_0^{\pi/2} (\cos \theta)^k d\theta = 2I_k$ , so (7.5) is the same as

$$I_{2n}I_{2n+1} = \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+2}{2})} \frac{\Gamma(\frac{2n+2}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+3}{2})}$$
$$= \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})^2}{4\Gamma(\frac{2n+1}{2}+1)}$$
$$= \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})^2}{4\frac{2n+1}{2}\Gamma(\frac{2n+1}{2})}$$
$$= \frac{\Gamma(\frac{1}{2})^2}{2(2n+1)}.$$

By (7.5),  $\pi = \Gamma(1/2)^2$ . We saw in the fifth proof that  $\Gamma(1/2) = \sqrt{\pi}$  if and only if  $J = \sqrt{\pi}/2$ .

## 8. Eighth Proof: Stirling's Formula

Besides the integral formula  $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$  that we have been discussing, another place in mathematics where  $\sqrt{2\pi}$  appears is in Stirling's formula:

$$n! \sim \frac{n^n}{e^n} \sqrt{2\pi n}$$
 as  $n \to \infty$ .

In 1730 De Moivre proved  $n! \sim C(n^n/e^n)\sqrt{n}$  for some positive number C without being able to determine C. Stirling soon thereafter showed  $C = \sqrt{2\pi}$  and wound up having the whole formula named after him. We will show that determining that the constant C in Stirling's formula is  $\sqrt{2\pi}$  is equivalent to showing that  $J = \sqrt{\pi}/2$  (or, equivalently, that  $I = \sqrt{2\pi}$ ).

Applying (7.4) repeatedly,

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$
  
=  $\frac{(2n-1)(2n-3)}{(2n)(2n-2)} I_{2n-4}$   
:  
=  $\frac{(2n-1)(2n-3)(2n-5)\cdots(5)(3)(1)}{(2n)(2n-2)(2n-4)\cdots(6)(4)(2)} I_0.$ 

Inserting  $(2n-2)(2n-4)(2n-6)\cdots(6)(4)(2)$  in the top and bottom,

$$I_{2n} = \frac{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\cdots(6)(5)(4)(3)(2)(1)}{(2n)((2n-2)(2n-4)\cdots(6)(4)(2))^2} \frac{\pi}{2} = \frac{(2n-1)!}{2n(2^{n-1}(n-1)!)^2} \frac{\pi}{2}$$

Applying De Moivre's asymptotic formula  $n! \sim C(n/e)^n \sqrt{n}$ ,

$$I_{2n} \sim \frac{C((2n-1)/e)^{2n-1}\sqrt{2n-1}}{2n(2^{n-1}C((n-1)/e)^{n-1}\sqrt{n-1})^2} \frac{\pi}{2} = \frac{(2n-1)^{2n}\frac{1}{2n-1}\sqrt{2n-1}}{2n\cdot 2^{2(n-1)}Ce(n-1)^{2n}\frac{1}{(n-1)^2}(n-1)} \frac{\pi}{2}$$

as  $n \to \infty$ . For any  $a \in \mathbf{R}$ ,  $(1 + a/n)^n \to e^a$  as  $n \to \infty$ , so  $(n + a)^n \sim e^a n^n$ . Substituting this into the above formula with a = -1 and n replaced by 2n,

(8.1) 
$$I_{2n} \sim \frac{e^{-1}(2n)^{2n} \frac{1}{\sqrt{2n}}}{2n \cdot 2^{2(n-1)} Ce(e^{-1}n^n)^2 \frac{1}{n^2}n} \frac{\pi}{2} = \frac{\pi}{C\sqrt{2n}}$$

Since  $I_{k-1} \sim I_k$ , the outer terms in (7.3) are both asymptotic to  $\sqrt{n}I_{2n} \sim \pi/(C\sqrt{2})$  by (8.1). Therefore

$$\int_0^{\sqrt{n}} e^{-y^2} \,\mathrm{d}y \to \frac{\pi}{C\sqrt{2}}$$

as  $n \to \infty$ , so  $J = \pi/(C\sqrt{2})$ . Therefore  $C = \sqrt{2\pi}$  if and only if  $J = \sqrt{\pi}/2$ .

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## 9. NINTH PROOF: THE ORIGINAL PROOF

The original proof that  $J = \sqrt{\pi}/2$  is due to Laplace [8] in 1774. (An English translation of Laplace's article is mentioned in the bibliographic citation for [8], with preliminary comments on that article in [18].) He wanted to compute

(9.1) 
$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{-\log x}}$$

Setting  $y = \sqrt{-\log x}$ , this integral is  $2 \int_0^\infty e^{-y^2} dy = 2J$ , so we expect (9.1) to be  $\sqrt{\pi}$ . Laplace's starting point for evaluating (9.1) was a formula of Euler:

(9.2) 
$$\int_0^1 \frac{x^r \, \mathrm{d}x}{\sqrt{1-x^{2s}}} \int_0^1 \frac{x^{s+r} \, \mathrm{d}x}{\sqrt{1-x^{2s}}} = \frac{1}{s(r+1)} \frac{\pi}{2}$$

for positive r and s. (Laplace himself said this formula held "whatever be" r or s, but if s < 0 then the number under the square root is negative.) Accepting (9.2), let  $r \to 0$  in it to get

(9.3) 
$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^{2s}}} \int_0^1 \frac{x^s \,\mathrm{d}x}{\sqrt{1-x^{2s}}} = \frac{1}{s} \frac{\pi}{2}$$

Now let  $s \to 0$  in (9.3). Then  $1 - x^{2s} \sim -2s \log x$  by L'Hopital's rule, so (9.3) becomes

$$\left(\int_0^1 \frac{\mathrm{d}x}{\sqrt{-\log x}}\right)^2 = \pi$$

Thus (9.1) is  $\sqrt{\pi}$ .

Euler's formula (9.2) looks mysterious, but we have met it before. In the formula let  $x^s = \cos \theta$  with  $0 \le \theta \le \pi/2$ . Then  $x = (\cos \theta)^{1/s}$ , and after some calculations (9.2) turns into

(9.4) 
$$\int_0^{\pi/2} (\cos\theta)^{(r+1)/s-1} \,\mathrm{d}\theta \int_0^{\pi/2} (\cos\theta)^{(r+1)/s} \,\mathrm{d}\theta = \frac{1}{(r+1)/s} \frac{\pi}{2}.$$

We used the integral  $I_k = \int_0^{\pi/2} (\cos \theta)^k \, d\theta$  before when k is a nonnegative integer. This notation makes sense when k is any positive real number, and then (9.4) assumes the form  $I_{\alpha}I_{\alpha+1} = \frac{1}{\alpha+1}\frac{\pi}{2}$  for  $\alpha = (r+1)/s - 1$ , which is (7.5) with a possibly nonintegral index. Letting r = 0 and s = 1/(2n+1)in (9.4) recovers (7.5). Letting  $s \to 0$  in (9.3) corresponds to letting  $n \to \infty$  in (7.5), so the proof in Section 7 is in essence a more detailed version of Laplace's 1774 argument.

## 10. Tenth Proof: Residue Theorem

We will calculate  $\int_{-\infty}^{\infty} e^{-x^2/2} dx$  using contour integrals and the residue theorem. However, we can't just integrate  $e^{-z^2/2}$ , as this function has no poles. For a long time nobody knew how to handle this integral using contour integration. For instance, in 1914 Watson [20, p. 79] wrote "Cauchy's theorem cannot be employed to evaluate all definite integrals; thus  $\int_{0}^{\infty} e^{-x^2} dx$  has not been evaluated except by other methods." In the 1940s several contour integral solutions were published using awkward contours such as parallelograms [10], [12, Sect. 5] (see [2, Exer. 9, p. 113] for a recent appearance). Our approach will follow Kneser [6, p. 121] (see also [13, pp. 413–414] or [22]), using a rectangular contour and the function

$$\frac{e^{-z^2/2}}{1 - e^{-\sqrt{\pi}(1+i)z}}$$

This function comes out of nowhere, so our first task is to motivate the introduction of this function.

We seek a meromorphic function f(z) to integrate around the rectangular contour  $\gamma_R$  in the figure below, with vertices at -R, R, R+ib, and -R+ib, where b will be fixed and we let  $R \to \infty$ .



Suppose  $f(z) \to 0$  along the right and left sides of  $\gamma_R$  uniformly as  $R \to \infty$ . Then by applying the residue theorem and letting  $R \to \infty$ , we would obtain (if the integrals converge)

$$\int_{-\infty}^{\infty} f(x) \,\mathrm{d}x + \int_{\infty}^{-\infty} f(x+ib) \,\mathrm{d}x = 2\pi i \sum_{a} \operatorname{Res}_{z=a} f(z),$$

where the sum is over poles of f(z) with imaginary part between 0 and b. This is equivalent to

$$\int_{-\infty}^{\infty} (f(x) - f(x + ib)) \, \mathrm{d}x = 2\pi i \sum_{a} \operatorname{Res}_{z=a} f(z)$$

Therefore we want f(z) to satisfy

(10.1) 
$$f(z) - f(z+ib) = e^{-z^2/2},$$

where f(z) and b need to be determined.

Let's try  $f(z) = e^{-z^2/2}/d(z)$ , for an unknown denominator d(z) whose zeros are poles of f(z). We want f(z) to satisfy

(10.2) 
$$f(z) - f(z+\tau) = e^{-z^2/2}$$

for some  $\tau$  (which will not be purely imaginary, so (10.1) doesn't quite work, but (10.1) is only motivation). Substituting  $e^{-z^2/2}/d(z)$  for f(z) in (10.2) gives us

(10.3) 
$$e^{-z^2/2} \left( \frac{1}{d(z)} - \frac{e^{-\tau z - \tau^2/2}}{d(z+\tau)} \right) = e^{-z^2/2}$$

Suppose  $d(z + \tau) = d(z)$ . Then (10.3) implies

$$d(z) = 1 - e^{-\tau z - \tau^2/2},$$

and with this definition of d(z),  $e^{-z^2/2}/d(z)$  satisfies (10.2) if and only if  $e^{\tau^2} = 1$ , or equivalently  $\tau^2 \in 2\pi i \mathbf{Z}$ . The simplest nonzero solution is  $\tau = \sqrt{\pi}(1+i)$ . From now on this is the value of  $\tau$ , so  $e^{-\tau^2/2} = e^{-i\pi} = -1$  and  $d(z) = 1 + e^{-\tau z}$ . Set

$$f(z) = \frac{e^{-z^2/2}}{d(z)} = \frac{e^{-z^2/2}}{1 + e^{-\tau z}}$$

which is Kneser's function mentioned earlier. This function satisfies (10.2) and we henceforth ignore the motivation (10.1). Poles of f(z) are at odd integral multiples of  $\tau/2$ .

We will integrate this f(z) around the rectangular contour  $\gamma_R$  below, whose height is Im $(\tau)$ .



The poles of f(z) nearest the origin are plotted in the figure; they lie along the line y = x. The only pole of f(z) inside  $\gamma_R$  (for  $R > \sqrt{\pi/2}$ ) is at  $\tau/2$ , so by the residue theorem

$$\int_{\gamma_R} f(z) \, \mathrm{d}z = 2\pi i \operatorname{Res}_{z=\tau/2} f(z) = 2\pi i \frac{e^{-\tau^2/8}}{(-\tau)e^{-\tau^2/2}} = \frac{2\pi i e^{3\tau^2/8}}{-\sqrt{\pi}(1+i)} = \frac{2\pi i e^{3\pi i/4}}{-\sqrt{\pi}(1+i)} = \sqrt{2\pi}.$$

Since the left and right sides of  $\gamma_R$  have the same length,  $\sqrt{\pi}$ , for all R, to show the integral of f along those sides tends to 0 uniformly as  $R \to \infty$ , it suffices to show  $f(z) \to 0$  uniformly along those sides as  $R \to \infty$ . Parametrize z along the left and right sides as -R + it and R + it with t running over  $[0, \sqrt{\pi}]$  in one direction or the other (which won't matter since we'll be taking absolute values). Then, using the reverse triangle inequality in the denominator, when  $R > \sqrt{\pi}$  (so R > t)

$$|f(R+it)| = \frac{|e^{-R^2/2 - iRt + t^2/2}|}{|1 + e^{-\tau(R+it)}|} \le \frac{e^{-R^2/2} e^{t^2/2}}{|1 - e^{-\operatorname{Re}(\tau(R+it))}|} \le \frac{e^{-R^2/2} e^{\pi/2}}{1 - e^{-\sqrt{\pi}(R-t)}} < \frac{e^{-R^2/2} e^{\pi/2}}{1 - e^{-\sqrt{\pi}(R-\sqrt{\pi})}},$$

which tends to 0 as  $R \to \infty$ . Also

$$|f(-R+it)| = \frac{|e^{-R^2/2 + iRt + t^2/2}|}{|1 + e^{-\tau(-R+it)}|} \le \frac{e^{-R^2/2}e^{t^2/2}}{|1 - e^{-\operatorname{Re}(\tau(-R+it))}|} \le \frac{e^{-R^2/2}e^{\pi/2}}{e^{\sqrt{\pi}(R+t)} - 1} < \frac{e^{-R^2/2}e^{\pi/2}}{e^{\sqrt{\pi}R} - 1}$$

which tends to 0 as  $R \to \infty$ . Thus

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} f(x) \,\mathrm{d}x + \int_{\infty+i\sqrt{\pi}}^{-\infty+i\sqrt{\pi}} f(z) \,\mathrm{d}z = \int_{-\infty}^{\infty} f(x) \,\mathrm{d}x - \int_{-\infty}^{\infty} f(x+i\sqrt{\pi}) \,\mathrm{d}x.$$

In the second integral, write  $i\sqrt{\pi}$  as  $\tau - \pi$  and use (real) translation invariance of dx to obtain

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} f(x) \,\mathrm{d}x - \int_{-\infty}^{\infty} f(x+\tau) \,\mathrm{d}x = \int_{-\infty}^{\infty} (f(x) - f(x+\tau)) \,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2/2} \,\mathrm{d}x \quad \text{by (10.2)}.$$

## 11. Eleventh Proof: Fourier transforms

For a continuous function  $f: \mathbf{R} \to \mathbf{C}$  that is rapidly decreasing at  $\pm \infty$ , its Fourier transform is the function  $\mathcal{F}f: \mathbf{R} \to \mathbf{C}$  defined by

(11.1) 
$$(\mathcal{F}f)(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy} \,\mathrm{d}x.$$

For example,  $(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x.$ 

Here are three properties of the Fourier transform.

• If f is differentiable, then after using differentiation under the integral sign on the Fourier transform of f we obtain

$$(\mathcal{F}f)'(y) = \int_{-\infty}^{\infty} -ixf(x)e^{-ixy} \,\mathrm{d}x = -i(\mathcal{F}(xf(x)))(y).$$

• Using integration by parts on the Fourier transform of f, with u = f(x) and  $dv = e^{-ixy} dx$ , we obtain

$$(\mathcal{F}(f'))(y) = iy(\mathcal{F}f)(y).$$

• If we apply the Fourier transform twice then we recover the original function up to interior and exterior scaling:

(11.2) 
$$(\mathcal{F}^2 f)(x) = 2\pi f(-x)$$

The  $2\pi$  is admittedly a nonobvious scaling factor here, and the proof of (11.2) is nontrivial. We'll show the appearance of  $2\pi$  in (11.2) is equivalent to the evaluation of I as  $\sqrt{2\pi}$ .

Fixing a > 0, set  $f(x) = e^{-ax^2}$ , so

$$f'(x) = -2axf(x)$$

Applying the Fourier transform to both sides of this equation implies  $iy(\mathcal{F}f)(y) = -2a\frac{1}{-i}(\mathcal{F}f)'(y)$ , which simplifies to  $(\mathcal{F}f)'(y) = -\frac{1}{2a}y(\mathcal{F}f)(y)$ . The general solution of  $g'(y) = -\frac{1}{2a}yg(y)$  is  $g(y) = -\frac{1}{2a}yg(y)$  $Ce^{-y^2/(4a)}$ , so

$$f(x) = e^{-ax^2} \Longrightarrow (\mathcal{F}f)(y) = Ce^{-y^2/(4a)}$$

for some constant C. We have 1/(4a) = a when a = 1/2, so set a = 1/2: if  $f(x) = e^{-x^2/2}$  then

(11.3) 
$$(\mathcal{F}f)(y) = Ce^{-y^2/2} = Cf(y).$$

Setting y = 0 in (11.3), the left side is  $(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} e^{-x^2/2} dx = I$ , so I = Cf(0) = C. Applying the Fourier transform to both sides of (11.3) with C = I and using (11.2), we get

 $2\pi f(-x) = I(\mathcal{F}f)(x) = I^2 f(x)$ . At x = 0 this becomes  $2\pi = I^2$ , so  $I = \sqrt{2\pi}$  since I > 0. That is the Gaussian integral calculation. If we didn't know that the constant on the right side of (11.2) is  $2\pi$ , whatever its value is would wind up being  $I^2$ , so saying  $2\pi$  appears on the right side of (11.2) is equivalent to saying  $I = \sqrt{2\pi}$ .

There are other ways to define the Fourier transform besides (11.1), such as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixy} \, \mathrm{d}x \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} \, \mathrm{d}x$$

These transforms have properties similar to the transform as defined in (11.1), so they can be used in its place to compute the Gaussian integral. Let's see how such a proof looks using the second alternative definition, which we'll write as

$$(\widetilde{\mathcal{F}}f)(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy} \,\mathrm{d}x$$

For this Fourier transform, the analogue of the three properties above for  $\mathcal{F}$  are

- $(\widetilde{\mathcal{F}}f)'(y) = -2\pi i (\widetilde{\mathcal{F}}(xf(x)))(y).$
- $(\widetilde{\mathcal{F}}(f'))(y) = 2\pi i y (\widetilde{\widetilde{\mathcal{F}}}f)(y).$   $(\widetilde{\mathcal{F}}^2 f)(x) = f(-x).$

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The last property for  $\widetilde{\mathcal{F}}$  looks nicer than that for  $\mathcal{F}$ , since there is no overall  $2\pi$ -factor on the right side (it has been hidden in the definition of  $\widetilde{\mathcal{F}}$ ). On the other hand, the first two properties for  $\widetilde{\mathcal{F}}$ have overall factors of  $2\pi$  on the right side while the first two properties of  $\mathcal{F}$  do not. You can't escape a role for  $\pi$  or  $2\pi$  somewhere in every possible definition of a Fourier transform.

Now let's run through the proof again with  $\widetilde{\mathcal{F}}$  in place of  $\mathcal{F}$ . For a > 0, set  $f(x) = e^{-ax^2}$ . Applying  $\widetilde{\mathcal{F}}$  to both sides of the equation f'(x) = -2axf(x),  $2\pi iy(\widetilde{\mathcal{F}}f)(y) = -2a\frac{1}{-(2\pi i)}(\mathcal{F}f)'(y)$ , and that is equivalent to  $(\widetilde{\mathcal{F}}f)'(y) = -\frac{2\pi^2}{a}y(\mathcal{F}f)(y)$ . Solutions of  $g'(y) = -\frac{2\pi^2}{a}yg(y)$  all look like  $Ce^{-(\pi^2/a)y^2}$ , so

$$f(x) = e^{-ax^2} \Longrightarrow (\widetilde{\mathcal{F}}f)(y) = Ce^{-(\pi^2/a)y^2}$$

for a constant C. We want  $\pi^2/a = \pi$  so that  $e^{-(\pi^2/a)y^2} = e^{-\pi y^2} = f(y)$ , which occurs for  $a = \pi$ . Thus when  $f(x) = e^{-\pi x^2}$  we have

(11.4) 
$$(\widetilde{\mathcal{F}}f)(y) = Ce^{-\pi y^2} = Cf(y).$$

When y = 0 in (11.4), this becomes  $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = C$ , so C = K: see the top of the first page for the definition of K as the integral of  $e^{-\pi x^2}$  over **R**.

Applying  $\widetilde{\mathcal{F}}$  to both sides of (11.4) with C = K and using  $(\widetilde{\mathcal{F}}^2 f)(x) = f(-x)$ , we get  $f(-x) = K(\widetilde{\mathcal{F}}f)(x) = K^2 f(x)$ . At x = 0 this becomes  $1 = K^2$ , so K = 1 since K > 0. That K = 1, or in more explicit form  $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$ , is equivalent to the evaluation of the Gaussian integral I with the change of variables  $y = \sqrt{2\pi x}$  in the integral for K.

## 12. HISTORY OF THE GAUSSIAN INTEGRAL

The function  $e^{-x^2/2}$ , or in the form  $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$  ("normal distribution") to have total integral 1 over **R**, plays an essential role in probability and statistics, and it was in probabilistic settings that it was first found. The approximation of a binomial distribution with many samples by a normal distribution, which is a mainstay of probability courses today, is how the normal distribution was first found in work of De Moivre in 1733. This role as a mere approximation did not make it stand out. In the 1770s

## Appendix A. Redoing Section 2 without improper integrals in Fubini's theorem

In this appendix we will work out the calculation of the Gaussian integral in Section 2 without relying on Fubini's theorem for improper integrals. The key equation is (2.1), which we recall:

$$\int_0^\infty \left( \int_0^\infty y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y = \int_0^\infty \left( \int_0^\infty y e^{-(t^2+1)y^2} \, \mathrm{d}y \right) \, \mathrm{d}t.$$

The calculation in Section 2 that the iterated integral on the right is  $\pi/4$  does not need Fubini's theorem in any form. It is going from the iterated integral on the left to  $\pi/4$  that used Fubini's theorem for improper integrals. The next theorem could be used as a substitute, and its proof will only use Fubini's theorem for integrals on rectangles.

Theorem A.1. For b > 1 and c > 1,

$$\int_0^\infty \left( \int_0^\infty y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y = \frac{\pi}{4} + O\left(\frac{1}{b}\right) + O\left(\frac{1}{\sqrt{c}}\right).$$

Having  $b \to \infty$  and  $c \to \infty$  in Theorem A.1 makes the right side  $\pi/4$  without changing the left side.

Lemma A.2. (1) For all 
$$x \in \mathbf{R}$$
,  $e^{-x^2} \le \frac{1}{x^2 + 1}$ .  
(2) For  $a > 0$   $\int_0^\infty \frac{\mathrm{d}x}{a^2 x^2 + 1} = \frac{\pi}{2a}$ .  
(3) For  $a > 0$  and  $c > 0$ ,  $\int_c^\infty \frac{\mathrm{d}x}{a^2 x^2 + 1} = \frac{1}{a} \left(\frac{\pi}{2} - \arctan(ac)\right)$ .  
(4) For  $a > 0$  and  $c > 0$ ,  $\int_c^\infty \frac{\mathrm{d}x}{a^2 x^2 + 1} < \frac{1}{a^2 c}$ .  
(5) For  $a > 0$ ,  $\frac{\pi}{2} - \arctan a < \frac{1}{a}$ .

*Proof.* The proofs of (1), (2), and (3) are left to the reader. To prove (4), replace  $1 + a^2t^2$  by the smaller value  $a^2t^2$ . To prove (5), write the difference as  $\int_a^\infty dx/(x^2+1)$  and then bound  $1/(x^2+1)$  above by  $1/x^2$ .

Now we prove Theorem A.1.

*Proof.* Step 1. For b > 1 and c > 1, we'll show the improper integral can be truncated to an integral over  $[0, \overline{b}] \times [0, c]$  plus error terms:

$$\int_0^\infty \left( \int_0^\infty y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y = \int_0^b \left( \int_0^c y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y + O\left(\frac{1}{\sqrt{c}}\right) + O\left(\frac{1}{b}\right).$$

Subtract the integral on the right from the integral on the left and split the outer integral  $\int_0^\infty$  into  $\int_0^b + \int_b^\infty$ :

$$\int_0^\infty \left( \int_0^\infty y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y - \int_0^b \left( \int_0^c y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y = \int_0^b \left( \int_c^\infty y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y + \int_b^\infty \left( \int_0^\infty y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y.$$

On the right side, we will show the first iterated integral is  $O(1/\sqrt{c})$  and the second iterated integral is O(1/b). The second iterated integral is simpler:

$$\begin{split} \int_{b}^{\infty} \left( \int_{0}^{\infty} y e^{-(t^{2}+1)y^{2}} dt \right) dy &= \int_{b}^{\infty} \left( \int_{0}^{\infty} e^{-(yt)^{2}} dt \right) y e^{-y^{2}} dy \\ &\leq \int_{b}^{\infty} \left( \int_{0}^{\infty} \frac{dt}{y^{2}t^{2}+1} \right) y e^{-y^{2}} dy \quad \text{by Lemma A.2(1)} \\ &= \int_{b}^{\infty} \frac{\pi}{2y} y e^{-y^{2}} dy \quad \text{by Lemma A.2(2)} \\ &= \frac{\pi}{2} \int_{b}^{\infty} e^{-y^{2}} dy \\ &\leq \frac{\pi}{2} \int_{b}^{\infty} \frac{dy}{y^{2}+1} \quad \text{by Lemma A.2(1)} \\ &= \frac{\pi}{2b} \quad \text{since } \frac{1}{y^{2}+1} < \frac{1}{y^{2}}, \end{split}$$

and this is O(1/b). Returning to the first iterated integral,

$$\begin{split} \int_{0}^{b} \left( \int_{c}^{\infty} y e^{-(t^{2}+1)y^{2}} dt \right) dy &= \int_{0}^{b} \left( \int_{c}^{\infty} e^{-(yt)^{2}} dt \right) y e^{-y^{2}} dy \\ &\leq \int_{0}^{b} \left( \int_{c}^{\infty} \frac{dt}{y^{2}t^{2}+1} \right) y e^{-y^{2}} dy \quad \text{by Lemma A.2(1)} \\ &= \int_{0}^{1} \left( \int_{c}^{\infty} \frac{dt}{y^{2}t^{2}+1} \right) y e^{-y^{2}} dy + \int_{1}^{b} \left( \int_{c}^{\infty} \frac{dt}{y^{2}t^{2}+1} \right) y e^{-y^{2}} dy \\ &\leq \int_{0}^{1} \left( \int_{c}^{\infty} \frac{dt}{y^{2}t^{2}+1} \right) y e^{-y^{2}} dy + \int_{1}^{b} \frac{1}{y^{2}c} y e^{-y^{2}} dy \quad \text{by Lemma A.2(4)} \\ &= \int_{0}^{1} \left( \frac{\pi}{2} - \arctan(yc) \right) e^{-y^{2}} dy + \frac{1}{c} \int_{1}^{b} \frac{dy}{y e^{y^{2}}} \quad \text{by Lemma A.2(3)} \\ &\leq \int_{0}^{1} \left( \frac{\pi}{2} - \arctan(yc) \right) dy + \frac{1}{c} \int_{1}^{\infty} \frac{dy}{y e^{y^{2}}}. \end{split}$$

The last term is O(1/c). We will show the first term is  $O(1/\sqrt{c})$  by carefully splitting up  $\int_0^1 dt dt dt$ . For  $0 < \varepsilon < 1$ ,

$$\int_0^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \, \mathrm{d}y = \int_0^\varepsilon \left(\frac{\pi}{2} - \arctan(yc)\right) \, \mathrm{d}y + \int_\varepsilon^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \, \mathrm{d}y.$$

Both integrals are positive, and the first one is less than  $(\pi/2)\varepsilon$ . The integrand of the second integral is less than 1/(yc) by Lemma A.2(5), so

$$\int_{\varepsilon}^{1} \left(\frac{\pi}{2} - \arctan(yc)\right) \, \mathrm{d}y < \int_{\varepsilon}^{1} \frac{\mathrm{d}y}{yc} < \frac{1-\varepsilon}{\varepsilon c} < \frac{1}{\varepsilon c}.$$

Therefore

$$0 < \int_0^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \, \mathrm{d}y < \frac{\pi}{2}\varepsilon + \frac{1}{\varepsilon c}$$

for each  $\varepsilon$  in (0,1). Use  $\varepsilon = 1/\sqrt{c}$  to get

$$0 < \int_0^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \, \mathrm{d}y < \frac{\pi}{2\sqrt{c}} + \frac{1}{\sqrt{c}} = O\left(\frac{1}{\sqrt{c}}\right).$$

That proves the first iterated integral is  $O(1/\sqrt{c}) + O(1/c) = O(1/\sqrt{c})$  as  $c \to \infty$ . Step 2. For b > 0 and c > 0, we will show

$$\int_0^b \left( \int_0^c y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y = \frac{\pi}{4} + O\left(\frac{1}{e^{b^2}}\right) + O\left(\frac{1}{c}\right).$$

By Fubini's theorem for continuous functions on a *rectangle* in  $\mathbf{R}^2$ ,

$$\int_0^b \left( \int_0^c y e^{-(t^2+1)y^2} \, \mathrm{d}t \right) \, \mathrm{d}y = \int_0^c \left( \int_0^b y e^{-(t^2+1)y^2} \, \mathrm{d}y \right) \, \mathrm{d}t.$$

For the inner integral on the right, the formula  $\int_0^b y e^{-ay^2} dy = 1/(2a) - 1/(2ae^{ab^2})$  for a > 0 tells us

$$\int_0^b y e^{-(t^2+1)y^2} \, \mathrm{d}y = \frac{1}{2(t^2+1)} - \frac{1}{2(t^2+1)e^{(t^2+1)b^2}},$$

SO

(A.1) 
$$\int_0^c \left( \int_0^b y e^{-(t^2+1)y^2} \, \mathrm{d}y \right) \, \mathrm{d}t = \frac{1}{2} \int_0^c \frac{\mathrm{d}t}{t^2+1} - \frac{1}{2} \int_0^c \frac{\mathrm{d}t}{(t^2+1)e^{(t^2+1)b^2}} \\ = \frac{1}{2} \arctan(c) - \frac{1}{2} \int_0^c \frac{\mathrm{d}t}{(t^2+1)e^{(t^2+1)b^2}}.$$

Let's estimate these last two terms. Since

$$\arctan(c) = \int_0^\infty \frac{\mathrm{d}t}{t^2 + 1} - \int_c^\infty \frac{\mathrm{d}t}{t^2 + 1} = \frac{\pi}{2} + O\left(\int_c^\infty \frac{\mathrm{d}t}{t^2}\right) = \frac{\pi}{2} + O\left(\frac{1}{c}\right)$$

and

$$\int_0^c \frac{\mathrm{d}t}{(t^2+1)e^{(t^2+1)b^2}} \le \int_0^c \frac{\mathrm{d}t}{t^2+1} \frac{1}{e^{b^2}} \le \int_0^\infty \frac{\mathrm{d}t}{t^2+1} \frac{1}{e^{b^2}} = O\left(\frac{1}{e^{b^2}}\right),$$

feeding these error estimates into (A.1) finishes Step 2.

#### References

- [1] D. Bell, "Poisson's remarkable calculation a method or a trick?" *Elem. Math.* 65 (2010), 29–36.
- [2] C. A. Berenstein and R. Gay, *Complex Variables*, Springer-Verlag, New York, 1991. [3] A. L. Delgado, "A Calculation of  $\int_0^\infty e^{-x^2} dx$ ," The College Math. J. **34** (2003), 321–323.
- [4] H. Iwasawa, "Gaussian Integral Puzzle," Math. Intelligencer **31** (2009), 38–41.
- [5] T. P. Jameson, "The Probability Integral by Volume of Revolution," Mathematical Gazette 78 (1994), 339–340.
- [6] H. Kneser, Funktionentheorie, Vandenhoeck and Ruprecht, 1958.
- [7] P. S. Laplace, *Théorie Analytique des Probabilités*, Courcier, 1812.
- [8] P. S. Laplace, "Mémoire sur la probabilité des causes par les évènemens," Oeuvres Complétes 8, 27–65. (English trans. by S. Stigler as "Memoir on the Probability of Causes of Events," Statistical Science 1 (1986), 364–378.)
- [9] P. M. Lee, http://www.york.ac.uk/depts/maths/histstat/normal history.pdf.
- [10] L. Mirsky, The Probability Integral, Math. Gazette 33 (1949), 279. URL http://www.jstor.org/stable/ 3611303.
- [11] C. P. Nicholas and R. C. Yates, "The Probability Integral," Amer. Math. Monthly 57 (1950), 412–413.
- [12] G. Polya, "Remarks on Computing the Probability Integral in One and Two Dimensions," pp. 63–78 in Berkeley Symp. on Math. Statist. and Prob., Univ. California Press, 1949.
- [13] R. Remmert, Theory of Complex Functions, Springer-Verlag, 1991.
- [14] M. Rozman, "Evaluate Gaussian integral using differentiation under the integral sign," Course notes for Physics 2400 (UConn), Spring 2016.
- [15] W. Rudin, Principles of Mathematical Analysis, 3rd ed., McGraw-Hill, 1976.
- [16] M. Spivak, Calculus, W. A. Benjamin, 1967.
- [17] S. Stahl, "The Evolution of the Normal Distribution," Math. Mag. 79 (2006), 96-113. URL https://maa.org/ sites/default/files/pdf/upload library/22/Allendoerfer/stahl96.pdf.
- [18] S. Stigler, "Laplace's 1774 Memoir on Inverse Probability," Statistical Science 1 (1986), 359–363.
- [19] J. van Yzeren, "Moivre's and Fresnel's Integrals by Simple Integration," Amer. Math. Monthly 86 (1979), 690-693.
- [20] G. N. Watson, Complex Integration and Cauchy's Theorem, Cambridge Univ. Press, Cambridge, 1914.
- [21] http://gowers.wordpress.com/2007/10/04/when-are-two-proofs-essentially-the-same/#comment-239.
- [22] http://math.stackexchange.com/questions/34767/int-infty-infty-e-x2-dx-with-complex-analysis.
- [23] http://math.stackexchange.com/questions/390850/integrating-int-infty-0-e-x2-dx-using-feynmansparametrization-trick