THE GAUSSIAN INTEGRAL

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Let

\[ I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, dx, \quad J = \int_0^{\infty} e^{-x^2} \, dx, \quad \text{and} \quad K = \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx. \]

These positive numbers are related: \( J = I / (2\sqrt{2}) \) and \( K = I / \sqrt{2\pi} \).

**Theorem.** With notation as above, \( I = \sqrt{2\pi} \), or equivalently \( J = \sqrt{\pi/2} \), or equivalently \( K = 1 \).

We will give multiple proofs of this. (Other lists of proofs are in [4] and [9].) It is subtle since \( e^{-\frac{1}{2}x^2} \) has no simple antiderivative. For comparison, \( \int_0^{\infty} x e^{-\frac{1}{2}x^2} \, dx \) can be computed with the antiderivative \( -e^{-\frac{1}{2}x^2} \) and equals 1.

1. **First Proof: Polar coordinates**

   The most widely known proof, due to Poisson [9, p. 3], expresses \( J^2 \) as a double integral and then uses polar coordinates. To start, write \( J^2 \) as an iterated integral using single-variable calculus:

   \[
   J^2 = J \int_0^{\infty} e^{-y^2} \, dy = \int_0^{\infty} J e^{-y^2} \, dy = \int_0^{\infty} \left( \int_0^{\infty} e^{-x^2} \, dx \right) e^{-y^2} \, dy = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} \, dx \, dy.
   \]

   View this as a double integral over the first quadrant. To compute it with polar coordinates, the first quadrant is \( \{(r,\theta) : r \geq 0 \text{ and } 0 \leq \theta \leq \pi/2\} \). Writing \( x^2 + y^2 \) as \( r^2 \) and \( dx \, dy \) as \( r \, dr \, d\theta \),

   \[
   J^2 = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} \, r \, dr \, d\theta
   \]

   \[
   = \int_0^{\infty} r e^{-r^2} \, dr \cdot \int_0^{\pi/2} d\theta
   \]

   \[
   = -\frac{1}{2} e^{-r^2} \bigg|_0^{\infty} \cdot \frac{\pi}{2}
   \]

   \[
   = \frac{1}{2} \cdot \frac{\pi}{2}
   \]

   \[
   = \frac{\pi}{4}.
   \]

   Since \( J > 0 \), \( J = \sqrt{\pi/2} \). It is argued in [1] that this method can’t be used on any other integral.

2. **Second Proof: Another change of variables**

   Our next proof uses another change of variables to compute \( J^2 \). As before,

   \[
   J^2 = \int_0^{\infty} \left( \int_0^{\infty} e^{-(x^2+y^2)} \, dx \right) \, dy.
   \]

   For a visualization of this calculation as a volume, in terms of \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) instead of \( J \), see https://www.youtube.com/watch?v=cy8r7WSuTII. We’ll do a volume calculation for \( I^2 \) in Section 5.
Instead of using polar coordinates, set $x = yt$ in the inner integral ($y$ is fixed). Then $dx = y\, dt$ and
\begin{equation}
J^2 = \int_0^\infty \left( \int_0^\infty e^{-y^2(t^2+1)}\, y\, dt \right) \, dy = \int_0^\infty \left( \int_0^\infty ye^{-y^2(t^2+1)}\, dy \right) \, dt ,
\end{equation}
where the interchange of integrals is justified by Fubini’s theorem for improper Riemann integrals. (The appendix gives an approach using Fubini’s theorem for Riemann integrals on rectangles.) Since $\int_0^\infty ye^{-ay^2}\, dy = \frac{1}{2a}$ for $a > 0$, we have
\begin{equation}
J^2 = \int_0^\infty \frac{dt}{2(t^2 + 1)} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} ,
\end{equation}
so $J = \sqrt{\pi}/2$. This proof is due to Laplace [7, pp. 94–96] and historically precedes the widely used technique of the previous proof. We will see in Section 9 what Laplace’s first proof was.

3. Third Proof: Differentiating under the integral sign

For $t > 0$, set
\begin{equation}
A(t) = \left( \int_0^t e^{-x^2} \, dx \right)^2 .
\end{equation}
The integral we want to calculate is $A(\infty) = J^2$ and then take a square root.
Differentiating $A(t)$ with respect to $t$ and using the Fundamental Theorem of Calculus,
\begin{equation}
A’(t) = 2 \int_0^t e^{-x^2} \, dx \cdot (\varepsilon - t^2) = 2e^{-t^2} \int_0^t e^{-x^2} \, dx .
\end{equation}
Let $x = ty$, so
\begin{equation}
A’(t) = 2e^{-t^2} \int_0^1 te^{-t^2y^2} \, dy = \int_0^1 2te^{-(1+y^2)^2} \, dy .
\end{equation}
The function under the integral sign is easily antidifferentiated with respect to $t$:
\begin{equation}
A’(t) = \int_0^1 \frac{\partial}{\partial t} e^{-(1+y^2)t^2} \, dy = - \frac{d}{dt} \int_0^1 e^{-(1+y^2)t^2} \, dy .
\end{equation}
Letting
\begin{equation}
B(t) = \int_0^1 e^{-t^2(1+x^2)} \, dx ,
\end{equation}
we have $A’(t) = -B’(t)$ for all $t > 0$, so there is a constant $C$ such that
\begin{equation}
A(t) = -B(t) + C
\end{equation}
for all $t > 0$. To find $C$, we let $t \to 0^+$ in (3.1). The left side tends to $\left( \int_0^0 e^{-x^2} \, dx \right)^2 = 0$ while
the right side tends to $- \int_0^1 dx/(1 + x^2) + C = -\pi/4 + C$. Thus $C = \pi/4$, so (3.1) becomes
\begin{equation}
\left( \int_0^t e^{-x^2} \, dx \right)^2 = \frac{\pi}{4} - \int_0^1 e^{-t^2(1+x^2)} \, dx .
\end{equation}
Letting $t \to \infty$ in this equation, we obtain $J^2 = \pi/4$, so $J = \sqrt{\pi}/2$.
A comparison of this proof with the first proof is in [20].
4. Fourth Proof: Another differentiation under the integral sign

Here is a second approach to finding $J$ by differentiation under the integral sign. I heard about it from Michael Rozman [14], who modified an idea on math.stackexchange [22], and in a slightly less elegant form it appeared much earlier in [18].

For $t \in \mathbb{R}$, set

$$F(t) = \int_{0}^{\infty} \frac{e^{-t^2(1+x^2)}}{1+x^2} \, dx.$$  

Then $F(0) = \int_{0}^{\infty} dx/(1 + x^2) = \pi/2$ and $F(\infty) = 0$. Differentiating under the integral sign,

$$F'(t) = \int_{0}^{\infty} -2te^{-t^2(1+x^2)} \, dx = -2te^{-t^2} \int_{0}^{\infty} e^{-(tx)^2} \, dx.$$  

Make the substitution $y = tx$, with $dy = t \, dx$, so

$$F'(t) = -2e^{-t^2} \int_{0}^{\infty} e^{-y^2} \, dy = -2Je^{-t^2}.$$  

For $b > 0$, integrate both sides from 0 to $b$ and use the Fundamental Theorem of Calculus:

$$\int_{0}^{b} F'(t) \, dt = -2J \int_{0}^{b} e^{-t^2} \, dt \implies F(b) - F(0) = -2J \int_{0}^{b} e^{-t^2} \, dt.$$  

Letting $b \to \infty$ in the last equation,

$$0 - \frac{\pi}{2} = -2J^2 \implies J^2 = \frac{\pi}{4} \implies J = \frac{\sqrt{\pi}}{2}.$$  

5. Fifth Proof: A volume integral

Our next proof is due to T. P. Jameson [5] and it was rediscovered by A. L. Delgado [3]. Revolve the curve $z = e^{-\frac{1}{2}x^2}$ in the $xz$-plane around the $z$-axis to produce the “bell surface” $z = e^{-\frac{1}{2}(x^2+y^2)}$. See below, where the $z$-axis is vertical and passes through the top point, the $x$-axis lies just under the surface through the point $0$ in front, and the $y$-axis lies just under the surface through the point $0$ on the left. We will compute the volume $V$ below the surface and above the $xy$-plane in two ways.

First we compute $V$ by horizontal slices, which are discs: $V = \int_{0}^{1} A(z) \, dz$ where $A(z)$ is the area of the disc formed by slicing the surface at height $z$. Writing the radius of the disc at height $z$ as $r(z)$, $A(z) = \pi r(z)^2$. To compute $r(z)$, the surface cuts the $xz$-plane at a pair of points $(x, e^{-\frac{1}{2}x^2})$ where the height is $z$, so $e^{-\frac{1}{2}x^2} = z$. Thus $x^2 = -2 \ln z$. Since $x$ is the distance of these points from the $z$-axis, $r(z)^2 = x^2 = -2 \ln z$, so $A(z) = \pi r(z)^2 = -2\pi \ln z$. Therefore

$$V = \int_{0}^{1} -2\pi \ln z \, dz = -2\pi (z \ln z - z) \bigg|_{0}^{1} = -2\pi (-1 - \lim_{z \to 0^+} z \ln z).$$  

By L’Hospital’s rule, $\lim_{z \to 0^+} z \ln z = 0$, so $V = 2\pi$. (A calculation of $V$ by shells is in [11].)

Next we compute the volume by vertical slices in planes $x = \text{constant}$. Vertical slices are scaled bell curves: look at the black contour lines in the picture. The equation of the bell curve along the top of the vertical slice with $x$-coordinate $x$ is $z = e^{-\frac{1}{2}(x^2+y^2)}$, where $y$ varies and $x$ is fixed. Then
V = \int_{-\infty}^{\infty} A(x) \, dx, \text{ where } A(x) \text{ is the area of the } x\text{-slice:}

A(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} \, dy = e^{-\frac{1}{2}x^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \, dy = e^{-\frac{1}{2}x^2} I.

Thus V = \int_{-\infty}^{\infty} A(x) \, dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} I \, dx = I \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, dx = I^2.

Comparing the two formulas for V, we have \(2\pi = I^2\), so \(I = \sqrt{2\pi}\).

6. Sixth Proof: The \(\Gamma\)-function

For any integer \(n \geq 0\), we have \(n! = \int_{0}^{\infty} t^n e^{-t} \, dt\). For \(x > 0\) we define

\[\Gamma(x) = \int_{0}^{\infty} t^x e^{-t} \frac{dt}{t},\]

so \(\Gamma(n) = (n-1)!\) when \(n \geq 1\). Using integration by parts, \(\Gamma(x + 1) = x\Gamma(x)\). One of the basic properties of the \(\Gamma\)-function [15, pp. 193–194] is

\[\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_{0}^{1} t^{x-1}(1-t)^{y-1} \, dt.\]
Set \( x = y = 1/2 \):

\[
\Gamma\left(\frac{1}{2}\right)^2 = \int_0^1 \frac{dt}{\sqrt{t(1-t)}}.
\]

Note

\[
\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} \ dt = \int_0^\infty e^{-t} \ dt = 2 \int_0^\infty e^{-2x} \ dx = 2J,
\]

so \( 4J^2 = \int_0^1 dt/\sqrt{t(1-t)} \). With the substitution \( t = \sin^2 \theta \),

\[
4J^2 = \int_0^{\pi/2} \frac{2 \sin \theta \cos \theta \ d\theta}{\sin \theta \cos \theta} = 2\frac{\pi}{2} = \pi,
\]

so \( J = \sqrt{\pi}/2 \). Equivalently, \( \Gamma(1/2) = \sqrt{\pi} \).

Any method that proves \( \Gamma(1/2) = \sqrt{\pi} \) is also a method that calculates \( \int_0^\infty e^{-x^2} \ dx \).

7. SEVENTH PROOF: ASYMPTOTIC ESTIMATES

We will show \( J = \sqrt{\pi}/2 \) by a technique whose steps are based on [16, p. 371].

For \( x \geq 0 \), power series expansions show \( 1 + x \leq e^x \leq 1/(1-x) \). Reciprocating and replacing \( x \) with \( x^2 \), we get

\[
1 - x^2 \leq e^{-x^2} \leq \frac{1}{1+x^2}.
\]

for all \( x \in \mathbb{R} \).

For any positive integer \( n \), raise the terms in (7.1) to the \( n \)th power and integrate from 0 to 1:

\[
\int_0^1 (1-x^2)^n \ dx \leq \int_0^1 e^{-nx^2} \ dx \leq \int_0^1 \frac{dx}{(1+x^2)^n}.
\]

Using the changes of variables \( x = \sin \theta \) on the left, \( x = y/\sqrt{n} \) in the middle, and \( x = \tan \theta \) on the right,

\[
\int_0^{\pi/2} (\cos \theta)^{2n+1} \ d\theta \leq \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} \ dy \leq \int_0^{\pi/4} (\cos \theta)^{2n-2} \ d\theta < \int_0^{\pi/2} (\cos \theta)^{2n-2} \ d\theta.
\]

Set \( I_k = \int_0^{\pi/2} (\cos \theta)^k \ d\theta \), so \( I_0 = \pi/2 \), \( I_1 = 1 \), and (7.2) implies

\[
\sqrt{n}I_{2n+1} \leq \int_0^{\sqrt{n}} e^{-y^2} \ dy < \sqrt{n}I_{2n-2}.
\]

We will show that as \( k \to \infty \), \( kI_k^2 \to \pi/2 \). Then

\[
\sqrt{n}I_{2n+1} = \frac{\sqrt{n}}{\sqrt{2n+1}} \sqrt{2n+1}I_{2n+1} \to \frac{1}{\sqrt{2}} \sqrt{\pi/2} = \frac{\sqrt{\pi}}{2}
\]

and

\[
\sqrt{n}I_{2n-2} = \frac{\sqrt{n}}{\sqrt{2n-2}} \sqrt{2n-2}I_{2n-2} \to \frac{1}{\sqrt{2}} \sqrt{\pi/2} = \frac{\sqrt{\pi}}{2},
\]

so by (7.3), \( \int_0^{\sqrt{n}} e^{-y^2} \ dy \to \sqrt{\pi}/2 \). Thus \( J = \sqrt{\pi}/2 \).
To show $kI_k^2 \to \pi/2$, first we compute several values of $I_k$ explicitly by a recursion. Using integration by parts,

$$I_k = \int_0^{\pi/2} (\cos \theta)^k \, d\theta = \int_0^{\pi/2} (\cos \theta)^{k-1} \cos \theta \, d\theta = (k - 1)(I_{k-2} - I_k),$$

so

(7.4)$$I_k = \frac{k - 1}{k} I_{k-2}.$$

Using (7.4) and the initial values $I_0 = \pi/2$ and $I_1 = 1$, the first few values of $I_k$ are computed and listed in Table 1.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$I_k$</th>
<th>$k$</th>
<th>$I_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\pi/2$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$(1/2)(\pi/2)$</td>
<td>3</td>
<td>$2/3$</td>
</tr>
<tr>
<td>4</td>
<td>$(3/8)(\pi/2)$</td>
<td>5</td>
<td>$8/15$</td>
</tr>
<tr>
<td>6</td>
<td>$(15/48)(\pi/2)$</td>
<td>7</td>
<td>$48/105$</td>
</tr>
</tbody>
</table>

Table 1.

From Table 1 we see that

(7.5)$$I_{2n}I_{2n+1} = \frac{1}{2n+1} \frac{\pi}{2}$$

for $0 \leq n \leq 3$, and this can be proved for all $n$ by induction using (7.4). Since $0 \leq \cos \theta \leq 1$ for $\theta \in [0, \pi/2]$, we have $I_k \leq I_{k-1} \leq I_{k-2} = \frac{k}{k-1}I_k$ by (7.4), so $I_{k-1} \sim I_k$ as $k \to \infty$. Therefore (7.5) implies

$$I_{2n}^2 \sim \frac{1}{2n} \frac{\pi}{2} \implies (2n)I_{2n}^2 \to \frac{\pi}{2}$$

as $n \to \infty$. Then

$$(2n + 1)I_{2n+1}^2 \sim (2n)I_{2n}^2 \to \frac{\pi}{2}$$

as $n \to \infty$, so $kI_k^2 \to \pi/2$ as $k \to \infty$. This completes our proof that $J = \sqrt{\pi}/2$.

**Remark 7.1.** This proof is closely related to the fifth proof using the $\Gamma$-function. Indeed, by (6.1)

$$\frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+1}{2} + \frac{1}{2}\right)} = \int_0^1 t^{(k+1)/2+1}(1-t)^{1/2-1} \, dt,$$

and with the change of variables $t = (\cos \theta)^2$ for $0 \leq \theta \leq \pi/2$, the integral on the right is equal to $2 \int_0^{\pi/2}(\cos \theta)^k \, d\theta = 2I_k$, so (7.5) is the same as

$$I_{2n}I_{2n+1} = \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2n+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2n+3}{2}\right)}$$

and

$$= \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)^2}{4\Gamma\left(\frac{2n+1}{2} + 1\right)}$$

$$= \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)^2}{4\Gamma\left(\frac{2n}{2} + 1\right)\Gamma\left(\frac{2n+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)^2}{2(2n + 1)}.$$
By (7.5), \(\pi = \Gamma(1/2)^2\). We saw in the fifth proof that \(\Gamma(1/2) = \sqrt{\pi}\) if and only if \(J = \sqrt{\pi}/2\).

8. Eighth Proof: Stirling’s Formula

Besides the integral formula \(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, dx = \sqrt{2\pi}\) that we have been discussing, another place in mathematics where \(\sqrt{2\pi}\) appears is in Stirling’s formula:

\[n! \sim \frac{n^n}{e^n} \sqrt{2\pi n} \quad \text{as} \quad n \to \infty.\]

In 1730 De Moivre proved \(n! \sim C(n^n/e^n)\sqrt{n}\) for some positive number \(C\) without being able to determine \(C\). Stirling soon thereafter showed \(C = \sqrt{2\pi}\) and wound up having the whole formula named after him. We will show that determining that the constant \(C\) in Stirling’s formula is \(\sqrt{2\pi}\) is equivalent to showing that \(J = \sqrt{\pi}/2\) (or, equivalently, that \(I = \sqrt{2\pi}\)).

Applying (7.4) repeatedly,

\[I_{2n} = \frac{2n - 1}{2n} I_{2n-2} = \frac{(2n - 1)(2n - 3)}{(2n)(2n - 2)} I_{2n-4} :: \frac{(2n - 1)(2n - 3)(2n - 5) \cdots (5)(3)(1)}{(2n)(2n - 2)(2n - 4) \cdots (6)(4)(2)} I_0.\]

Inserting \((2n - 2)(2n - 4)(2n - 6) \cdots (6)(4)(2)\) in the top and bottom,

\[I_{2n} = \frac{(2n - 1)(2n - 2)(2n - 3)(2n - 4)(2n - 5) \cdots (6)(5)(4)(3)(2)(1) \pi}{(2n)((2n - 2)(2n - 4) \cdots (6)(4)(2))^2} = \frac{(2n - 1)\pi}{2n(2n-1(n-1)!^2/2}.\]

Applying De Moivre’s asymptotic formula \(n! \sim C(n/e)^n \sqrt{n},\)

\[I_{2n} \sim \frac{C((2n - 1)/e)^{2n-1}\sqrt{2n - 1}}{2n(2n-1)C((n - 1)/e)^{n-1}\sqrt{n - 1}} \frac{\pi}{2} = \frac{(2n - 1)^2 \pi}{2n \cdot 2^{2(n-1)}Ce(n - 1)^2n (n - 1)^2\pi}.\]

as \(n \to \infty\). For any \(a \in \mathbb{R}, (1 + a/n)^n \to e^a\) as \(n \to \infty\), so \((n + a)^n \sim e^a n^a\). Substituting this into the above formula with \(a = -1\) and \(n\) replaced by \(2n,\)

\[(8.1) \quad I_{2n} \sim \frac{e^{-1}(2n)^2 \pi}{2n \cdot 2^{2(n-1)}Ce(e^{-1}n^2)^2 \pi n^2} = \frac{\pi}{C \sqrt{2n}}.\]

Since \(I_{k-1} \sim I_k\), the outer terms in (7.3) are both asymptotic to \(\sqrt{n}I_{2n} \sim \pi/(C\sqrt{2})\) by (8.1). Therefore

\[\int_0^{\sqrt{n}} e^{-y^2} \, dy \to \frac{\pi}{C \sqrt{2}}\]

as \(n \to \infty\), so \(J = \pi/(C\sqrt{2})\). Therefore \(C = \sqrt{2\pi}\) if and only if \(J = \sqrt{\pi}/2\).
9. Ninth Proof: The original proof

The original proof that \( J = \sqrt{\pi}/2 \) is due to Laplace [8] in 1774. (An English translation of Laplace’s article is mentioned in the bibliographic citation for [8], with preliminary comments on that article in [17].) He wanted to compute

\[
\int_0^1 \frac{dx}{\sqrt{-\log x}}.
\]

Setting \( y = \sqrt{-\log x} \), this integral is \( 2 \int_0^\infty e^{-y^2} \, dy = 2J \), so we expect (9.1) to be \( \sqrt{\pi} \).

Laplace’s starting point for evaluating (9.1) was a formula of Euler:

\[
\int_0^1 \frac{x^r}{\sqrt{1-x^2}} \, dx \int_0^1 \frac{x^{s+r}}{\sqrt{1-x^{2s}}} \, dx = \frac{1}{s(r+1)} \frac{\pi}{2}
\]

for positive \( r \) and \( s \). (Laplace himself said this formula held “whatever be” \( r \) or \( s \), but if \( s < 0 \) then the number under the square root is negative.) Accepting (9.2), let \( r \to 0 \) in it to get

\[
\int_0^1 \frac{dx}{\sqrt{1-x^{2s}}} \int_0^1 \frac{x^s \, dx}{\sqrt{1-x^{2s}}} = \frac{1}{s} \frac{\pi}{2}.
\]

Now let \( s \to 0 \) in (9.3). Then \( 1-x^{2s} \sim -2s \log x \) by L’Hôpital’s rule, so (9.3) becomes

\[
\left( \int_0^1 \frac{dx}{\sqrt{-\log x}} \right)^2 = \pi.
\]

Thus (9.1) is \( \sqrt{\pi} \).

Euler’s formula (9.2) looks mysterious, but we have met it before. In the formula let \( x^s = \cos \theta \) with \( 0 \leq \theta \leq \pi/2 \). Then \( x = (\cos \theta)^{1/s} \), and after some calculations (9.2) turns into

\[
\int_0^{\pi/2} (\cos \theta)^{(r+1)/s-1} \, d\theta \int_0^{\pi/2} (\cos \theta)^{(r+1)/s} \, d\theta = \frac{1}{(r+1)/s} \frac{\pi}{2}.
\]

We used the integral \( I_k = \int_0^{\pi/2} (\cos \theta)^k \, d\theta \) before when \( k \) is a nonnegative integer. This notation makes sense when \( k \) is any positive real number, and then (9.4) assumes the form \( I_\alpha I_{\alpha+1} = \frac{1}{\alpha+1} \frac{\pi}{2} \) for \( \alpha = (r+1)/s - 1 \), which is (7.5) with a possibly nonintegral index. Letting \( r = 0 \) and \( s = 1/(2n+1) \) in (9.4) recovers (7.5). Letting \( s \to 0 \) in (9.3) corresponds to letting \( n \to \infty \) in (7.5), so the proof in Section 7 is in essence a more detailed version of Laplace’s 1774 argument.

10. Tenth Proof: Residue theorem

We will calculate \( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \) using contour integrals and the residue theorem. However, we can’t just integrate \( e^{-z^2/2} \), as this function has no poles. For a long time nobody knew how to handle this integral using contour integration. For instance, in 1914 Watson [19, p. 79] wrote “Cauchy’s theorem cannot be employed to evaluate all definite integrals; thus \( \int_0^{\infty} e^{-x^2} \, dx \) has not been evaluated except by other methods.” In the 1940s several contour integral solutions were published using awkward contours such as parallelograms [10], [12, Sect. 5] (see [2, Exer. 9, p. 113] for a recent appearance). Our approach will follow Kneser [6, p. 121] (see also [13, pp. 413–414] or [21]), using a rectangular contour and the function

\[
e^{-z^2/2} \frac{1}{1-e^{-\sqrt{\pi}(1+i)z}}.
\]
This function comes out of nowhere, so our first task is to motivate the introduction of this function.

We seek a meromorphic function $f(z)$ to integrate around the rectangular contour $\gamma_R$ in the figure below, with vertices at $-R$, $R$, $R+ib$, and $-R+ib$, where $b$ will be fixed and we let $R \to \infty$.

![Diagram of rectangular contour](image)

Suppose $f(z) \to 0$ along the right and left sides of $\gamma_R$ uniformly as $R \to \infty$. Then by applying the residue theorem and letting $R \to \infty$, we would obtain (if the integrals converge)

$$
\int_{-\infty}^{\infty} f(x) \, dx + \int_{-\infty}^{\infty} f(x + ib) \, dx = 2\pi i \sum_a \text{Res}_{z=a} f(z),
$$

where the sum is over poles of $f(z)$ with imaginary part between 0 and $b$. This is equivalent to

$$
\int_{-\infty}^{\infty} (f(x) - f(x + ib)) \, dx = 2\pi i \sum_a \text{Res}_{z=a} f(z).
$$

Therefore we want $f(z)$ to satisfy

(10.1) \quad f(z) - f(z + ib) = e^{-z^2/2},

where $f(z)$ and $b$ need to be determined.

Let’s try $f(z) = e^{-z^2/2}/d(z)$, for an unknown denominator $d(z)$ whose zeros are poles of $f(z)$. We want $f(z)$ to satisfy

(10.2) \quad f(z) - f(z + \tau) = e^{-z^2/2}

for some $\tau$ (which will not be purely imaginary, so (10.1) doesn’t quite work, but (10.1) is only motivation). Substituting $e^{-z^2/2}/d(z)$ for $f(z)$ in (10.2) gives us

(10.3) \quad e^{-z^2/2} \left( \frac{1}{d(z)} - \frac{e^{-\tau z - \tau^2/2}}{d(z + \tau)} \right) = e^{-z^2/2}.

Suppose $d(z + \tau) = d(z)$. Then (10.3) implies

$$
d(z) = 1 - e^{-\tau z - \tau^2/2},
$$

and with this definition of $d(z)$, $e^{-z^2/2}/d(z)$ satisfies (10.2) if and only if $e^{\tau^2} = 1$, or equivalently $\tau^2 \in 2\pi i \mathbb{Z}$. The simplest nonzero solution is $\tau = \sqrt{\pi}(1 + i)$. From now on this is the value of $\tau$, so $e^{-\tau^2/2} = e^{-i\pi} = -1$ and we set

$$
f(z) = \frac{e^{-z^2/2}}{d(z)} = \frac{e^{-z^2/2}}{1 + e^{-\tau z}},
$$

which is Kneser’s function mentioned earlier. This function satisfies (10.2) and we henceforth ignore the motivation (10.1). Poles of $f(z)$ are at odd integral multiples of $\tau/2$.

We will integrate this $f(z)$ around the rectangular contour $\gamma_R$ below, whose height is $\text{Im}(\tau)$.
The poles of \( f(z) \) nearest the origin are plotted in the figure; they lie along the line \( y = x \). The only pole of \( f(z) \) inside \( \gamma_R \) (for \( R > \sqrt{\pi}/2 \)) is at \( \tau/2 \), so by the residue theorem

\[
\int_{\gamma_R} f(z) \, dz = 2\pi i \text{Res}_{z=\tau/2} f(z) = 2\pi i \frac{e^{-\tau^2/8}}{(-\tau)e^{-\tau^2/2}} = \frac{2\pi i e^{3\tau^2/8}}{-\sqrt{\pi}(1+i)} = \sqrt{2} \pi.
\]

As \( R \to \infty \), the value of \( |f(z)| \) tends to 0 uniformly along the left and right sides of \( \gamma_R \), so

\[
\sqrt{2} \pi = \int_{-\infty}^{\infty} f(x) \, dx + \int_{-\infty+i\sqrt{\pi}}^{\infty+i\sqrt{\pi}} f(z) \, dz \]

\[
= \int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} f(x+i\sqrt{\pi}) \, dx.
\]

In the second integral, write \( i\sqrt{\pi} \) as \( \tau - \pi \) and use (real) translation invariance of \( \, dx \) to obtain

\[
\sqrt{2} \pi = \int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} f(x+\tau) \, dx
\]

\[
= \int_{-\infty}^{\infty} (f(x) - f(x+\tau)) \, dx
\]

\[
= \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \quad \text{by (10.2).}
\]

11. Eleventh Proof: Fourier transforms

For a continuous function \( f : \mathbb{R} \to \mathbb{C} \) that is rapidly decreasing at \( \pm \infty \), its Fourier transform is the function \( \mathcal{F}f : \mathbb{R} \to \mathbb{C} \) defined by

\[
(\mathcal{F}f)(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy} \, dx.
\]

For example, \( (\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(x) \, dx \).

Here are three properties of the Fourier transform.

- If \( f \) is differentiable, then after using differentiation under the integral sign on the Fourier transform of \( f \) we obtain

\[
(\mathcal{F}f)'(y) = \int_{-\infty}^{\infty} -ixf(x)e^{-ixy} \, dx = -i(\mathcal{F}(xf))(y).
\]
Applying 2π escape a role for π
The last property for ˜F
For this Fourier transform, the analogue of the three properties above for ˜F looks nicer than that for F. On the other hand, the first two properties for ˜F have overall factors of 2π on the right side while the first two properties of F do not. You can’t escape a role for π or 2π somewhere in every possible definition of a Fourier transform.
Now let’s run through the proof again with ˜F in place of F. For a > 0, set f(x) = e−ax². Applying ˜F to both sides of the equation f′(x) = −2axf(x), 2πiy(˜Ff)(y) = −2a−1/(2πi)(Ff)′(y),
The calculation in Section 2 that the iterated integral on the right is \( \pi^2 / a \).

Proof. The proofs of (1), (2), and (3) are left to the reader. To prove (4), replace 1 + \( \pi^2 / a \) with the smaller value \( a^2 \). To prove (5), write the difference as \( \int_{a^2}^{\infty} dx / (x^2 + 1) \) and then bound 1/(x^2 + 1) above by 1/x^2.

Now we prove Theorem A.1.

APPENDIX A. Redoing Section 2 without improper integrals in Fubini’s theorem

In this appendix we will work out the calculation of the Gaussian integral in Section 2 without relying on Fubini’s theorem for improper integrals. The key equation is (2.1), which we recall:

\[
\int_0^\infty \left( \int_0^\infty ye^{-(t^2+1)y^2} \, dt \right) \, dy = \int_0^\infty \left( \int_0^\infty ye^{-(t^2+1)y^2} \, dy \right) \, dt.
\]

The calculation in Section 2 that the iterated integral on the right is \( \pi / 4 \) does not need Fubini’s theorem in any form. It is going from the iterated integral on the left to \( \pi / 4 \) that used Fubini’s theorem for improper integrals. The next theorem could be used as a substitute, and its proof will only use Fubini’s theorem for integrals on rectangles.

Theorem A.1. For \( b > 1 \) and \( c > 1 \),

\[
\int_0^\infty \left( \int_0^\infty ye^{-(t^2+1)y^2} \, dt \right) \, dy = \frac{\pi}{4} + o\left( \frac{1}{b} \right) + o\left( \frac{1}{\sqrt{c}} \right).
\]

Having \( b \to \infty \) and \( c \to \infty \) in Theorem A.1 makes the right side \( \pi / 4 \) without changing the left side.

Lemma A.2.

1. For all \( x \in \mathbb{R} \), \( e^{-x^2} \leq \frac{1}{x^2 + 1} \).
2. For \( a > 0 \), \( \int_0^\infty \frac{dx}{a^2 x^2 + 1} = \frac{\pi}{2a} \).
3. For \( a > 0 \) and \( c > 0 \), \( \int_c^\infty \frac{dx}{a^2 x^2 + 1} = \frac{1}{a} \left( \frac{\pi}{2} - \text{arctan}(ac) \right) \).
4. For \( a > 0 \) and \( c > 0 \), \( \int_c^\infty \frac{dx}{a^2 x^2 + 1} < \frac{1}{a^2 c} \).
5. For \( a > 0 \), \( \frac{\pi}{2} - \text{arctan} a < \frac{1}{a} \).

Proof. The proofs of (1), (2), and (3) are left to the reader. To prove (4), replace 1 + \( a^2 t^2 \) by the smaller value \( a^2 t^2 \). To prove (5), write the difference as \( \int_a^\infty dx / (x^2 + 1) \) and then bound 1/(x^2 + 1) above by 1/x^2. \( \square \)
Proof. Step 1. For \( b > 1 \) and \( c > 1 \), we’ll show the improper integral can be truncated to an integral over \([0, b] \times [0, c]\) plus error terms:

\[
\int_0^\infty \left( \int_0^\infty ye^{-(t^2+1)y^2} \, dt \right) \, dy = \int_0^b \left( \int_0^c ye^{-(t^2+1)y^2} \, dt \right) \, dy + O \left( \frac{1}{\sqrt{c}} \right) + O \left( \frac{1}{b} \right).
\]

Subtract the integral on the right from the integral on the left and split the outer integral \( \int_0^\infty \) into \( \int_0^b + \int_b^\infty \):

\[
\int_0^\infty \left( \int_0^\infty ye^{-(t^2+1)y^2} \, dt \right) \, dy - \int_0^b \left( \int_0^c ye^{-(t^2+1)y^2} \, dt \right) \, dy = \int_0^b \left( \int_c^\infty ye^{-(t^2+1)y^2} \, dt \right) \, dy
\]

\[
+ \int_b^\infty \left( \int_0^c ye^{-(t^2+1)y^2} \, dt \right) \, dy.
\]

On the right side, we will show the first iterated integral is \( O(1/\sqrt{c}) \) and the second iterated integral is \( O(1/b) \). The second iterated integral is simpler:

\[
\int_b^\infty \left( \int_0^\infty ye^{-(t^2+1)y^2} \, dt \right) \, dy = \int_b^\infty \left( \int_0^\infty e^{-(yt)^2} \, dt \right) ye^{-y^2} \, dy
\]

\[
\leq \int_b^\infty \left( \int_0^\infty \frac{dt}{y^2t^2 + 1} \right) ye^{-y^2} \, dy \quad \text{by Lemma A.2(1)}
\]

\[
= \int_b^\infty \frac{\pi}{2y} ye^{-y^2} \, dy \quad \text{by Lemma A.2(2)}
\]

\[
= \frac{\pi}{2} \int_b^\infty e^{-y^2} \, dy
\]

\[
\leq \frac{\pi}{2} \int_b^\infty \frac{dy}{y^2 + 1} \quad \text{by Lemma A.2(1)}
\]

\[
= \frac{\pi}{2b} \quad \text{since } \frac{1}{y^2 + 1} < \frac{1}{y^2},
\]

and this is \( O(1/b) \). Returning to the first iterated integral,

\[
\int_0^b \left( \int_c^\infty ye^{-(t^2+1)y^2} \, dt \right) \, dy = \int_0^b \left( \int_c^\infty e^{-(yt)^2} \, dt \right) ye^{-y^2} \, dy
\]

\[
\leq \int_0^b \left( \int_c^\infty \frac{dt}{y^2t^2 + 1} \right) ye^{-y^2} \, dy \quad \text{by Lemma A.2(1)}
\]

\[
= \int_0^1 \left( \int_c^\infty \frac{dt}{y^2t^2 + 1} \right) ye^{-y^2} \, dy + \int_1^b \left( \int_c^\infty \frac{dt}{y^2t^2 + 1} \right) ye^{-y^2} \, dy
\]

\[
\leq \int_0^1 \left( \int_c^\infty \frac{dt}{y^2t^2 + 1} \right) ye^{-y^2} \, dy + \int_1^b \frac{1}{y^2c} ye^{-y^2} \, dy \quad \text{by Lemma A.2(4)}
\]

\[
= \int_0^1 \left( \frac{\pi}{2} - \arctan(yc) \right) e^{-y^2} \, dy + \frac{1}{c} \int_1^b \frac{dy}{ye^{y^2}} \quad \text{by Lemma A.2(3)}
\]

\[
\leq \int_0^1 \left( \frac{\pi}{2} - \arctan(yc) \right) \, dy + \frac{1}{c} \int_1^\infty \frac{dy}{ye^{y^2}}.
\]

The last term is \( O(1/c) \). We will show the first term is \( O(1/\sqrt{c}) \) by carefully splitting up \( \int_0^1 \).
For $0 < \varepsilon < 1$,  
\[
\int_0^1 \left( \frac{\pi}{2} - \arctan(yc) \right) \, dy = \int_0^\varepsilon \left( \frac{\pi}{2} - \arctan(yc) \right) \, dy + \int_\varepsilon^1 \left( \frac{\pi}{2} - \arctan(yc) \right) \, dy.
\]
Both integrals are positive, and the first one is less than $(\pi/2)\varepsilon$. The integrand of the second integral is less than $1/(yc)$ by Lemma A.2(5), so  
\[
\int_\varepsilon^1 \left( \frac{\pi}{2} - \arctan(yc) \right) \, dy < \int_\varepsilon^1 \frac{dy}{yc} < \frac{1 - \varepsilon}{\varepsilon c} < \frac{1}{\varepsilon c}.
\]
Therefore  
\[
0 < \int_0^1 \left( \frac{\pi}{2} - \arctan(yc) \right) \, dy < \frac{\pi}{2} \varepsilon + \frac{1}{\varepsilon c}
\]
for each $\varepsilon$ in $(0,1)$. Use $\varepsilon = 1/\sqrt{c}$ to get  
\[
0 < \int_0^1 \left( \frac{\pi}{2} - \arctan(yc) \right) \, dy < \frac{\pi}{2} \sqrt{c} + \frac{1}{\sqrt{c}} = O \left( \frac{1}{\sqrt{c}} \right).
\]
That proves the first iterated integral is $O(1/\sqrt{c}) + O(1/c) = O(1/\sqrt{c})$ as $c \to \infty$.

**Step 2.** For $b > 0$ and $c > 0$, we will show  
\[
\int_0^b \left( \int_0^c ye^{-(t^2+1)y^2} \, dt \right) \, dy = \frac{\pi}{4} + O \left( \frac{1}{e^{b^2}} \right) + O \left( \frac{1}{c} \right).
\]
By Fubini’s theorem for continuous functions on a rectangle in $\mathbb{R}^2$,  
\[
\int_0^b \left( \int_0^c ye^{-(t^2+1)y^2} \, dt \right) \, dy = \int_0^c \left( \int_0^b ye^{-(t^2+1)y^2} \, dy \right) \, dt.
\]
For the inner integral on the right, the formula $\int_0^b ye^{-ay^2} \, dy = 1/(2a) - 1/(2ae^{ab^2})$ for $a > 0$ tells us  
\[
\int_0^b ye^{-(t^2+1)y^2} \, dy = \frac{1}{2(t^2 + 1)} - \frac{1}{2(t^2 + 1)e^{(t^2+1)b^2}},
\]
so  
\[
\int_0^c \left( \int_0^b ye^{-(t^2+1)y^2} \, dy \right) \, dt = \frac{1}{2} \int_0^c \frac{dt}{t^2 + 1} - \frac{1}{2} \int_0^c \frac{dt}{(t^2 + 1)e^{(t^2+1)b^2}}
\]
\[= \frac{1}{2} \arctan(c) - \frac{1}{2} \int_0^c \frac{dt}{(t^2 + 1)e^{(t^2+1)b^2}}. \tag{A.1}
\]
Let’s estimate these last two terms. Since  
\[
\arctan(c) = \int_0^\infty \frac{dt}{t^2 + 1} - \int_c^\infty \frac{dt}{t^2 + 1} = \frac{\pi}{2} + O \left( \int_c^\infty \frac{dt}{t^2} \right) = \frac{\pi}{2} + O \left( \frac{1}{c} \right)
\]
and  
\[
\int_0^c \frac{dt}{(t^2 + 1)e^{(t^2+1)b^2}} \leq \int_0^c \frac{dt}{t^2 + 1} \frac{1}{e^{b^2}} \leq \int_0^\infty \frac{dt}{t^2 + 1} \frac{1}{e^{b^2}} = O \left( \frac{1}{e^{b^2}} \right),
\]
feeding these error estimates into (A.1) finishes Step 2. □
THE GAUSSIAN INTEGRAL

REFERENCES