# ARC LENGTH, INTEGRATION BY PARTS, AND $\pi$

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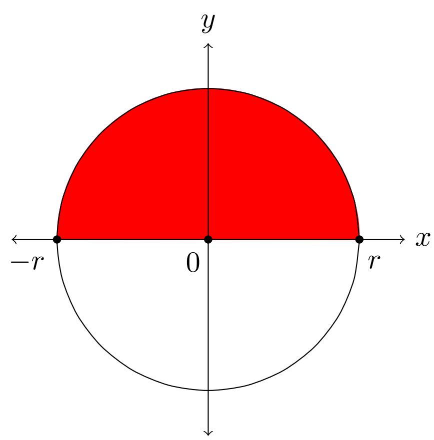
The number $\pi$ appears in two formulas about circles: their area and circumference. A circle of radius $r$ has area $\pi r^2$ and circumference $2\pi r$. Why does the *same* number $\pi$ show up in both formulas? It's not a surprise that $\pi$ occurs in the circumference formula, since that is essentially the way $\pi$ is defined when we first learn about $\pi$: the ratio of the circumference $C$ to the diameter $D$ for any circle is defined to be $\pi$, and the equation $\pi = C/D$ is the same as $C = \pi D = \pi(2r) = 2\pi r$.

To explain why $\pi$ is in the formula for a circle's area, we will write the area $A$ and circumference $C$ as definite integrals, using an arc length integral in the case of $C$. By definition $\pi = C/D = C/(2r)$ and we expect that also $\pi = A/r^2$, meaning we expect $C/(2r) = A/r^2$. Using *integration by parts* we will derive the equation $C/(2r) = A/r^2$, so from $C = 2\pi r$ we get $A = r^2(C/(2r)) = Cr/2 = \pi r^2$.

To express $C$ and $A$ as integrals, we will work with a half-circle, as shaded below. The area of the circle is twice the shaded region's area and the circumference of this circle is twice the arc length along the circle from $(-r, 0)$ to $(r, 0)$.




The area of the shaded region is the area under $y = \sqrt{r^2 - x^2}$ where $-r \le x \le r$, so

$$A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx. \tag{1}$$

The length of $y = \sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$ is $\int_{-r}^{r} \sqrt{1 + (dy/dx)^2}\,dx$:

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{r^2 - x^2}} = -\frac{x}{\sqrt{r^2 - x^2}} \Longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2 - x^2} \Longrightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2},$$

so

$$C = 2\int_{-r}^{r} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\int_{-r}^{r} \frac{r\,dx}{\sqrt{r^2 - x^2}}. \tag{2}$$

We can remove the dependence on $r$ in the integrals (1) and (2) by making the change of variables $x = rt$ where $-1 \le t \le 1$:

$$A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = A = 2r^2 \int_{-1}^{1} \sqrt{1-t^2}\,dt$$

and

$$C = 2\int_{-r}^{r} \frac{r\,dx}{\sqrt{r^2-x^2}} = 2r\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}.$$

Thus

$$\frac{C}{2r} = \frac{A}{r^2} \Longleftrightarrow \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = 2\int_{-1}^{1} \sqrt{1-t^2}\,dt.$$

To prove $C/(2r) = A/r^2$, we will prove that last equation:

$$\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \stackrel{?}{=} 2\int_{-1}^{1} \sqrt{1-t^2}\,dt. \tag{3}$$

(Note the integral on the left side is improper since the integrand has vertical asymptotes at $t = \pm 1$. We will not focus on that.) In the integral on the right side of (3) use integration by parts with

$$u = \sqrt{1-t^2} \quad \text{and} \quad dv = dt.$$

Then

$$du = \frac{-2t}{2\sqrt{1-t^2}}\,dt = -\frac{t}{\sqrt{1-t^2}}\,dt \quad \text{and} \quad v = t,$$

so

$$\int_{-1}^{1} \sqrt{1-t^2}\,dt = \int_{-1}^{1} u\,dv = \left. uv \right|_{-1}^{1} - \int_{-1}^{1} v\,du = 0 + \int_{-1}^{1} \frac{t^2}{\sqrt{1-t^2}}\,dt. \tag{4}$$

Simplify the last integrand by writing the numerator $t^2$ as $t^2 - 1 + 1$:

$$\begin{aligned}
\int_{-1}^{1} \frac{t^2}{\sqrt{1-t^2}}\,dt &= \int_{-1}^{1} \frac{t^2-1+1}{\sqrt{1-t^2}}\,dt \\
&= \int_{-1}^{1} \frac{t^2-1}{\sqrt{1-t^2}}\,dt + \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,dt \\
&= \int_{-1}^{1} -\frac{(1-t^2)}{\sqrt{1-t^2}}\,dt + \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,dt \\
&= -\int_{-1}^{1} \sqrt{1-t^2}\,dt + \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,dt,
\end{aligned}$$

and feeding this into (4) gives us

$$\int_{-1}^{1} \sqrt{1-t^2}\,dt = -\int_{-1}^{1} \sqrt{1-t^2}\,dt + \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,dt.$$

Adding $\int_{-1}^{1} \sqrt{1-t^2}\,dt$ to both sides gives us (3), which completes the proof that if we define $\pi$ so that a circle with radius $r$ has circumference $2\pi r$ then the area of that circle is $\pi r^2$. The two sides of (3) turn out to be two ways to describe $\pi$: the left side is the arc length of a half-circle of radius 1 and the right side is the area of a circle of radius 1.