The number \( \pi \) appears in formulas for two properties of a circle: its area and its circumference. A circle of radius \( r \) has area \( \pi r^2 \) and circumference \( 2\pi r \). Why does the same number \( \pi \) show up in both formulas? It’s not a surprise that \( \pi \) occurs in the circumference formula, since that is more or less the way \( \pi \) is defined when we first learn about it: the ratio of the circumference \( C \) to the diameter \( D = 2r \) for any circle is defined to be \( \pi \), and the equation \( \pi = C/D \) is the same as \( C = \pi D = \pi (2r) = 2\pi r \).

To explain why \( \pi \) shows up in the formula for the area of a circle as well, we will focus on a special case of circles with radius \( 1 \) (to keep the notation simpler), for which the circumference is \( 2\pi \) by definition and we want to show the area is \( \pi \). We will express the area \( A \) and circumference \( C \) of this circle as integrals, using the arc length integral in the case of \( C \), and then use integration by parts to show \( C = 2A \), so from \( C = 2\pi \) we get \( A = \pi \).

To express \( C \) and \( A \) as integrals, we will work with half a circle, as shaded below. The circle has area twice the shaded region and circumference equal to twice the arc length from \((1,0)\) to \((-1,0)\). Both of these can be expressed as integrals.

The area of the shaded region is the area under the circular arc from \( x = -1 \) to \( x = 1 \). The equation of the circular arc along the top of the shaded region is \( y = \sqrt{1-x^2} \), so

\[
A = 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx.
\]

The length of the circular arc along the top of the shaded region is the length of the curve \( y = \sqrt{1-x^2} \) from \( x = -1 \) to \( x = 1 \), which is \( \int_{-1}^{1} \sqrt{1+(dy/dx)^2} \, dx \). When \( y = \sqrt{1-x^2} \), we have

\[
\frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}} \implies \left( \frac{dy}{dx} \right)^2 = \frac{x^2}{1-x^2} \implies 1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2},
\]

so

\[
C = 2 \int_{-1}^{1} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = 2 \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}.
\]
Using (1) and (2),

\[ C = 2A \iff 2 \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 4 \int_{-1}^{1} \sqrt{1-x^2} \, dx \iff \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx. \]

To prove \( C = 2A \), we will prove that last equation:

(3) \[ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx. \]

In the integral on the right side of (3) use integration by parts with

\[ u = \sqrt{1-x^2} \quad \text{and} \quad dv = dx. \]

Then

\[ du = \frac{-2x}{2\sqrt{1-x^2}} \, dx = -\frac{x}{\sqrt{1-x^2}} \, dx \quad \text{and} \quad v = x, \]

so

(4) \[ \int_{-1}^{1} \sqrt{1-x^2} \, dx = \int_{-1}^{1} u \, dv = uv \bigg|_{-1}^{1} - \int_{-1}^{1} v \, du = 0 + \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} \, dx. \]

Simplify the last integrand by writing the numerator \( x^2 \) as \( x^2 - 1 + 1 \):

\[ \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} \, dx = \int_{-1}^{1} \frac{x^2 - 1 + 1}{\sqrt{1-x^2}} \, dx \]

\[ = \int_{-1}^{1} \frac{x^2 - 1}{\sqrt{1-x^2}} \, dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx \]

\[ = \int_{-1}^{1} (1-x^2) \, dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx \]

\[ = -\int_{-1}^{1} \sqrt{1-x^2} \, dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx, \]

and feeding this into (4) gives us

\[ \int_{-1}^{1} \sqrt{1-x^2} \, dx = -\int_{-1}^{1} \sqrt{1-x^2} \, dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx. \]

Adding \( \int_{-1}^{1} \sqrt{1-x^2} \, dx \) to both sides gives us (3), which completes the proof that if we define \( \pi \) so that a circle with radius 1 has circumference \( 2\pi \) then the area of a circle with radius 1 is \( \pi \).