ARC LENGTH, INTEGRATION BY PARTS, AND π

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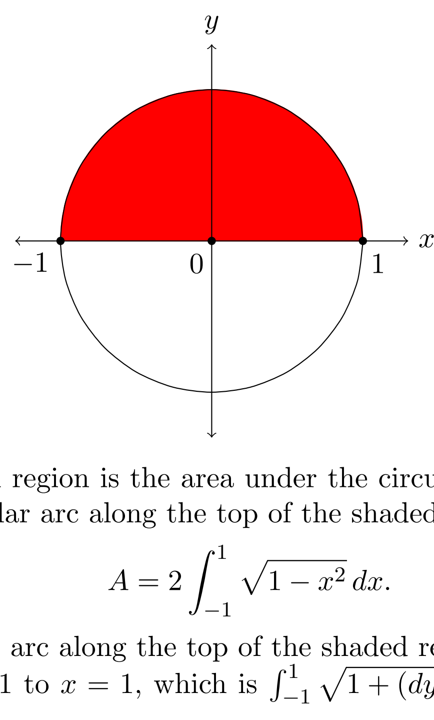
The number π appears in formulas for two properties of a circle: its area and its circumference. A circle of radius r has area πr^2 and circumference $2\pi r$. Why does the same number π show up in both formulas? It's not a surprise that π occurs in the circumference formula, since that is more or less the way π is defined when we first learn about it: the ratio of the circumference C to the diameter $D = 2r$ for any circle is defined to be π, and the equation $\pi = C/D$ is the same as $C = \pi D = \pi(2r) = 2\pi r$.

To explain why π shows up in the formula for the area of a circle as well, we will focus on a special case of circles with radius 1 (to keep the notation simpler), for which the circumference is 2π by definition and we want to show the area is π. We will express the area A and circumference C of this circle as integrals, using the arc length integral in the case of C, and then use *integration by parts* to show $C = 2A$, so from $C = 2\pi$ we get $A = \pi$.

To express C and A as integrals, we will work with half a circle, as shaded below. The circle has area twice the shaded region and circumference equal to twice the arc length from $(1, 0)$ to $(-1, 0)$. Both of these can be expressed as integrals.



The area of the shaded region is the area under the circular arc from $x = -1$ to $x = 1$. The equation of the circular arc along the top of the shaded region is $y = \sqrt{1 - x^2}$, so

$$A = 2\int_{-1}^{1} \sqrt{1 - x^2}\, dx. \tag{1}$$

The length of the circular arc along the top of the shaded region is the length of the curve $y = \sqrt{1 - x^2}$ from $x = -1$ to $x = 1$, which is $\int_{-1}^{1} \sqrt{1 + (dy/dx)^2}\, dx$. When $y = \sqrt{1 - x^2}$, we have

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}} \Longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1 - x^2} \Longrightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2},$$

so

$$C = 2\int_{-1}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}. \tag{2}$$

Using (1) and (2),

$$C = 2A \Longleftrightarrow 2\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 4\int_{-1}^{1} \sqrt{1-x^2}\,dx \Longleftrightarrow \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2\int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

To prove $C = 2A$, we will prove that last equation:

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \stackrel{?}{=} 2\int_{-1}^{1} \sqrt{1-x^2}\,dx. \tag{3}$$

In the integral on the right side of (3) use integration by parts with

$$u = \sqrt{1-x^2} \quad \text{and} \quad dv = dx.$$

Then

$$du = \frac{-2x}{2\sqrt{1-x^2}}\,dx = -\frac{x}{\sqrt{1-x^2}}\,dx \quad \text{and} \quad v = x,$$

so

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \int_{-1}^{1} u\,dv = \left. uv\right|_{-1}^{1} - \int_{-1}^{1} v\,du = 0 + \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx. \tag{4}$$

Simplify the last integrand by writing the numerator x^2 as $x^2 - 1 + 1$:

$$\begin{aligned}
\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int_{-1}^{1} \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx \\
&= \int_{-1}^{1} \frac{x^2-1}{\sqrt{1-x^2}}\,dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx \\
&= \int_{-1}^{1} -\frac{(1-x^2)}{\sqrt{1-x^2}}\,dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx \\
&= -\int_{-1}^{1} \sqrt{1-x^2}\,dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx,
\end{aligned}$$

and feeding this into (4) gives us

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = -\int_{-1}^{1} \sqrt{1-x^2}\,dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx.$$

Adding $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ to both sides gives us (3), which completes the proof that if we define π so that a circle with radius 1 has circumference 2π then the area of a circle with radius 1 is π.