# BOUNDARY BEHAVIOR OF POWER SERIES: ABEL'S THEOREM

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We want to discuss the two identities

$$\log(1+x) = \sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n}, \quad \sqrt{1+x} = \sum_{n\geq 0}\frac{(-1)^{n-1}(2n)!}{2^{2n}n!^2(2n-1)}x^n \tag{1}$$

at $x = 1$. In both equations, the series on the right side converges for $|x| < 1$ and the function on the left is defined for $|x| < 1$. Both sides of each equation satisfy the same first-order differential equation ($y' = 1/(1+x)$ on the left, $y' = \frac{1}{2y}$ on the right) and have the same values at $x = 0$, so the two sides are equal for $-1 < x < 1$. But what happens at the endpoints? For example, the series on the left in (1) converges at $x = 1$ since it is alternating there, but that alone doesn't prove the series at $x = 1$ is $\log 2$.

Let's be sure not to confuse two issues: convergence of the series at the boundary and knowing if the value of the series at the boundary equals what we think it should equal. How can we show the values of the series at the endpoints are what we expect by continuity from inside the interval of convergence? Rather than treating these two examples by special methods adapted just for them (see [1, p. 254] for that), we will explain a more systematic approach to boundary behavior of power series.

**Theorem 1** (Abel, 1826)**.** *Let $g(x) = \sum_{n\geq 0} c_n x^n$ be a power series which converges for $|x| < 1$. If $\sum_{n\geq 0} c_n$ converges then*

$$\lim_{x\to 1^-} g(x) = \sum_{n\geq 0} c_n.$$

In other words, *if* a power series converges at $x = 1$ then its value at $x = 1$ is the limit of its values at $x$ as $x \to 1^-$, so a power series has built-in continuity in its behavior.

Before we prove Abel's theorem, let's see how it applies to our previous examples at $x = 1$.

**Example 1.** In (1), set $g(x) = \sum_{n\geq 1}(-1)^{n-1}x^n/n$ for $|x| < 1$. Then $g(x) = \log(1 + x)$ for $|x| < 1$. The series $g(1)$ converges since it's alternating, so by Abel's theorem

$$g(1) = \lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} \log(1+x) = \log 2$$

since $\log(1 + x)$ is continuous on $[0, 1]$.

**Example 2.** In (1), set $g(x) = \sum_{n\geq 0}\frac{(-1)^{n-1}(2n)!}{2^{2n}n!^2(2n-1)}x^n$ for $|x| < 1$, so $g(x) = \sqrt{1+x}$ here. The series $g(1)$ is absolutely convergent, so by Abel's theorem and continuity of $\sqrt{1+x}$ on $[0, 1]$,

$$g(1) = \lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} \sqrt{1+x} = \sqrt{2}.$$

Abel's theorem says that if a power series converges on $(-1,1)$ and also at $x=1$ then its value at $x=1$ is determined by continuity from the left of 1. You must know the series converges at $x=1$ before you can apply Abel's theorem. This theorem does *not* say that if a function has a power series representation in $(-1,1)$ and has a limit as $x\to 1^-$ that the series converges at $x=1$ and equals the limiting value there. Here is a counterexample.

**Example 3.** Let $g(x)=1/(1+x^2)$, which is differentiable for all real $x$. When $|x|<1$, $g(x)=\sum_{n\geq 0}(-1)^n x^{2n}$ by expanding a geometric series. While $g(x)$ has a limit as $x\to 1^-$ (namely $1/2$), the power series does not converge at $x=1$.

Now we prove Abel's theorem. The main tool will be summation by parts: for two sequences $u_1,\dots,u_N$ and $v_0,\dots,v_N$,

$$\sum_{n=1}^{N} u_n(v_n-v_{n-1})=(u_Nv_N-u_1v_0)-\sum_{n=0}^{N-1} v_n(u_{n+1}-u_n).$$

*Proof.* We are given that $\sum_{n\geq 0} c_n x^n$ converges for $|x|<1$ and at $x=1$. Our goal is to prove

$$\lim_{x\to 1^-}\sum_{n\geq 0} c_n x^n=\sum_{n\geq 0} c_n.$$

For $-1<x<1$ we will work with the truncated sums $\sum_{n=0}^{N} c_n x^n$ and $\sum_{n=0}^{N} c_n$. Set

$$s_n=c_0+c_1+\cdots+c_n$$

for $n\geq 0$. Note $s_n-s_{n-1}=c_n$ for $n\geq 1$. Then

$$\begin{aligned}
\sum_{n=0}^{N} c_n x^n &= c_0+\sum_{n=1}^{N} x^n(s_n-s_{n-1})\\
&= c_0+\sum_{n=1}^{N} u_n(s_n-s_{n-1}) \text{ where } u_n=x^n\\
&= c_0+u_Ns_N-u_1s_0-\sum_{n=1}^{N-1} s_n(u_{n+1}-u_n) \text{ by summation by parts}\\
&= c_0+x^Ns_N-xc_0-\sum_{n=1}^{N-1} s_n(x^{n+1}-x^n)\\
&= (1-x)c_0+x^Ns_N+\sum_{n=1}^{N-1} s_n(x^n-x^{n+1})\\
&= (1-x)c_0+x^Ns_N+\sum_{n=1}^{N-1} s_n(1-x)x^n.
\end{aligned}$$

Since $s_0 = c_0$, we can absorb the first term into the sum as the $n = 0$ term, and then pull a $1 - x$ out of each term in the sum:

$$\sum_{n=0}^{N} c_n x^n = x^N s_N + (1-x) \sum_{n=0}^{N-1} s_n x^n. \tag{2}$$

By hypothesis, the left side of (2) converges as $N \to \infty$. Also $x^N s_N \to 0$ as $N \to \infty$ since $x^N \to 0$ and $s_N$ is bounded (in fact $s_N$ converges since we're assuming $\sum_{n=1}^{\infty} c_n$ converges). Therefore when $-1 < x < 1$ the partial sums on the right side of (2) converge as $N \to \infty$ and we get

$$\sum_{n \geq 0} c_n x^n = (1-x) \sum_{n \geq 0} s_n x^n. \tag{3}$$

Let $s = \sum_{n \geq 0} c_n$. We want to show $\sum_{n \geq 0} c_n x^n \to s$ as $x \to 1^-$. We subtract $s$ from both sides of (3) and write

$$\sum_{n \geq 0} c_n x^n - s = (1-x) \sum_{n \geq 0} (s_n - s) x^n, \tag{4}$$

where on the right side we used the formula $(1-x) \sum_{n \geq 0} x^n = 1$. Our goal is to show the right side of (4) tends to 0 as $x \to 1^-$.

By assumption, $s_n \to s$ as $n \to \infty$. Pick a positive number $\varepsilon$. For all large $n$, say $n \geq M$, $|s_n - s| \leq \varepsilon$. Then we break up the right side of (4) into two sums

$$\sum_{n \geq 0} c_n x^n - s = (1-x) \sum_{n=0}^{M-1} (s_n - s) x^n + (1-x) \sum_{n \geq M} (s_n - s) x^n$$

and estimate:

$$\begin{aligned} \left| \sum_{n \geq 0} c_n x^n - s \right| &\leq |1-x| \sum_{n=0}^{M-1} |s_n - s||x|^n + |1-x| \sum_{n \geq M} |s_n - s||x|^n \\ &\leq |1-x| \sum_{n=0}^{M-1} |s_n - s||x|^n + |1-x| \sum_{n \geq M} \varepsilon |x|^n \\ &= |1-x| \sum_{n=0}^{M-1} |s_n - s||x|^n + |1-x| \varepsilon \frac{|x|^M}{1-|x|} \\ &< |1-x| \sum_{n=0}^{M-1} |s_n - s||x|^n + |1-x| \varepsilon \frac{1}{1-|x|}. \end{aligned}$$

Taking $0 < x < 1$, $|1-x| = 1-x$ so this upper bound becomes

$$\left| \sum_{n \geq 0} c_n x^n - s \right| < |1-x| \sum_{n=0}^{M-1} |s_n - s| + \varepsilon. \tag{5}$$

When $x \to 1^-$, the first term on the right side of (5) tends to 0 on account of the $1-x$ there. (Note the upper index of summation $M-1$ has nothing to do with $x$, so it does not change as $x \to 1^-$.) When $x$ is close enough to 1, we can make the first term on the right side at most $\varepsilon$, so

$$\left|\sum_{n\geq 0} c_n x^n - s\right| \leq \varepsilon + \varepsilon = 2\varepsilon \tag{6}$$

as $x \to 1^-$. Since $\varepsilon$ is an arbitrary positive number, the left side of (6) must go to zero as $x \to 1^-$. $\square$

Abel's theorem was stated for the behavior of series at the point $x = 1$, but rescaling lets us apply it to other points, as follows.

**Corollary 2.** *Suppose a power series $\sum_{n\geq 0} c_n x^n$ converges for $|x| < r$. If the series converges at $r$ or $-r$ then the value of the series there is the limit of the values of the series as $x$ tends to the endpoint from inside the interval. That is,*

(a) *if $\sum_{n\geq 0} c_n r^n$ converges then*

$$\lim_{x\to r^-} \sum_{n\geq 0} c_n x^n = \sum_{n\geq 0} c_n r^n,$$

(b) *if $\sum_{n\geq 0} c_n (-r)^n$ converges then*

$$\lim_{x\to -r^+} \sum_{n\geq 0} c_n x^n = \sum_{n\geq 0} c_n (-r)^n.$$

*Proof.* In (a), let $a_n = c_n r^n$ and $g(x) = \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} c_n (rx)^n$ for $|x| < 1$. This series converges at $x = 1$ so Abel's theorem tells us

$$\sum_{n\geq 0} a_n = \lim_{x\to 1^-} \sum_{n\geq 0} a_n x^n = \lim_{x\to 1^-} \sum_{n\geq 0} c_n r^n x^n = \lim_{x\to r^-} \sum_{n\geq 0} c_n x^n,$$

where the limit changed from $x \to 1^-$ to $x \to r^-$ in the last equation (by replacing $x$ with $x/r$). Since $a_n = c_n r^n$, the left side is $\sum_{n\geq 0} a_n = \sum_{n\geq 0} c_n r^n$.

The argument in (b) is similar, using $a_n = c_n(-r)^n$. $\square$

**Example 4.** In the second equation in (1), the series on the right is (absolutely) convergent at $x = -1$. The function $\sqrt{1+x}$ is continuous from the right at $x = -1$, with value 0, so by Corollary 2 the second series in (1) at $x = -1$ has value 0:

$$0 = -\sum_{n\geq 0} \frac{(2n)!}{2^{2n} n!^2 (2n-1)} = 1 - \frac{1}{2} - \frac{1}{8} - \frac{1}{16} - \frac{5}{128} - \frac{7}{256} - \cdots,$$

or equivalently

$$1 = \sum_{n\geq 1} \frac{(2n)!}{2^{2n} n!^2 (2n-1)}.$$

## References

[1] E. Hairer, G. Wanner, *Analysis by Its History*, Springer-Verlag, New York, 1996.