THE REMAINDER IN TAYLOR SERIES

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1. Introduction

Let $f(x)$ be infinitely differentiable on an interval I around a number a. On the interval I, *Taylor's inequality* bounds the difference between $f(x)$ and its nth degree Taylor polynomial centered at a

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

in terms of the magnitude of the $(n+1)$th derivative of f: if $|f^{(n+1)}(x)| \leq M$ for all x in I then Taylor's inequality says

$$\text{if } b \text{ is in } I \text{ then } |f(b) - T_{n,a}(b)| \leq M\frac{|b-a|^{n+1}}{(n+1)!}.$$

This is easy if $b = a$ since $f(b) - T_{n,b}(b) = 0$. For $b \neq a$, we will derive Taylor's inequality in two ways using an *exact* formula for $f(x) - T_{n,a}(x)$ involving derivatives or integrals.

Theorem 1.1 (Differential form of the remainder (Lagrange, 1797))**.** *With notation as above, for $n \geq 0$ and b in the interval I with $b \neq a$,*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} = T_{n,a}(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some c strictly between a and b.

The number c depends on a, b, and n. When $n = 0$ the theorem says $f(b) = f(a) + f'(c)(b-a)$ for some c strictly between a and b. That is the Mean Value Theorem.

Theorem 1.2 (Integral form of the remainder (Cauchy, 1821))**.** *With notation as above, for $n \geq 0$ and b in the interval I with $b \neq a$,*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{f^{(n+1)}(t)}{n!}(b-t)^n\,dt = T_{n,a}(b) + \int_a^b \frac{f^{(n+1)}(t)}{n!}(b-t)^n\,dt.$$

If $n = 0$ this is $f(b) = f(a) + \int_a^b f'(t)\,dt$, which is the Fundamental Theorem of Calculus. Unlike the differential form of the remainder, the integral form of the remainder involves no additional parameters like c.

2. Differential (Lagrange) form of the remainder

To prove Theorem 1.1 we will use Rolle's theorem. Recall this theorem says if F is continuous on $[a,b]$, differentiable on (a,b), and $F(a) = F(b)$ then $F'(c) = 0$ where c is strictly between a and b.

Proof of Theorem 1.1. The following argument is based on a comment by Pieter-Jan De Smet on the page https://gowers.wordpress.com/2014/02/11/taylors-theorem-with-the-lagrange- form-of-the-remainder/.

There is a number C such that

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{C}{(n+1)!}(b-a)^{n+1} \tag{2.1}$$

since we can solve this equation for C (the factor $(b-a)^{n+1}$ is nonzero). We want to show $C = f^{(n+1)}(c)$ for some c strictly between a and b, and will do this by replacing a everywhere in (2.1) with a variable x to which we'll be able to use Rolle's theorem.

Consider the function

$$E(x) = f(b) - \left(\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{C}{(n+1)!}(b-x)^{n+1}\right),$$

for which $E(a) = 0$ by the choice of C and $E(b) = f(b) - f(b) = 0$. Then $E'(c) = 0$ for some c between a and b by Rolle's theorem. Let's now compute $E'(x)$ to see there is a lot of cancellation in it!

Using the product rule and some algebra,

$$\begin{aligned}
E'(x) &= -\left(\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{C}{(n+1)!}(b-x)^{n+1}\right)' \\
&= -\left(f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{C}{(n+1)!}(b-x)^{n+1}\right)' \\
&= -\left(f'(x) + \sum_{k=1}^{n}\left(\frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1}\right) - \frac{C}{n!}(b-x)^n\right) \\
&= -\left(f'(x) + \sum_{k=1}^{n} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1} - \frac{C}{n!}(b-x)^n\right) \\
&= -\left(f'(x) + \sum_{k=1}^{n} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{C}{n!}(b-x)^n\right) \\
&= -\left(f'(x) + \frac{f^{(n+1)}(x)}{n!}(b-x)^n - f'(x) - \frac{C}{n!}(b-x)^n\right) \\
&= \frac{-f^{(n+1)}(x) + C}{n!}(b-x)^n.
\end{aligned}$$

Therefore having $E'(c) = 0$ means $C = f^{(n+1)}(c)$, so (2.1) becomes

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}. \qquad \square$$

Taylor's inequality is an immediate consequence of this differential form of the remainder: if $|f^{(n+1)}(x)| \le M$ for all x from a to b, then $|f^{(n+1)}(c)| \le M$, so $|f(b) - T_{n,a}(b)| = |f^{(n+1)}(c)(b-a)^{n+1}/(n+1)!| \le M|b-a|^{n+1}/(n+1)!$.

3. Integral (Cauchy) form of the remainder

Proof of Theorem 1.2. Start with the Fundamental Theorem of Calculus in the form

$$f(b) = f(a) + \int_a^b f'(t)\,dt.$$

Apply integration by parts with $u = f'(t)$ and $dv = dt$, so $du = f''(t)\,dt$ and take $v = t - b$ (*not* $v = t$) to get

$$\begin{aligned} f(b) &= f(a) + f'(t)(t-b)\Big|_a^b - \int_a^b (t-b)f''(t)\,dt \\ &= f(a) - f'(a)(a-b) - \int_a^b (t-b)f''(t)\,dt \\ &= f(a) + f'(a)(b-a) + \int_a^b (b-t)f''(t)\,dt. \end{aligned}$$

Apply integration by parts again with $u = f''(t)$ and $dv = (b-t)\,dt$, so $du = f'''(t)\,dt$ and take $v = -(b-t)^2/2$. Then

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + \int_a^b (b-t)f''(t)\,dt \\ &= f(a) + f'(a)(b-a) - \frac{f''(t)}{2}(b-t)^2\Big|_a^b + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,dt \\ &= f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,dt. \end{aligned}$$

After another integration by parts with $u = f'''(t)$ and $dv = \frac{1}{2}(b-t)^2\,dt$ we get

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f'''(a)}{6}(b-a)^3 + \int_a^b \frac{(b-t)^3}{6}f^{(4)}(t)\,dt.$$

After repeated integration by parts we eventually get

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{(b-t)^n}{n!}f^{(n+1)}(t)\,dt. \qquad \square$$

We will derive Taylor's inequality from Theorem 1.2 in two ways.

Method 1: Assume $|f^{(n+1)}(t)| \leq M$ for all t from a to b. For $a < b$,

$$|f(b) - T_{n,a}(b)| \leq \int_a^b \frac{|b-t|^n}{n!} |f^{(n+1)}(t)| \, dt \leq \int_a^b \frac{(b-t)^n}{n!} M \, dt = M \frac{(b-a)^{n+1}}{(n+1)!},$$

where the last calculation comes from the Fundamental Theorem of Calculus. For $b < a$, we get in a similar way $|f(b) - T_{n,a}(b)| \leq M(a-b)^{n+1}/(n+1)!$. Putting both cases together,

$$|f^{(n+1)}(t)| \leq M \text{ for } t \text{ from } a \ \text{ to } b \Longrightarrow |f(b) - T_{n,a}(b)| \leq M \frac{|b-a|^{n+1}}{(n+1)!}.$$

Method 2: We make a change of variables in the integral to bypass the need for separate cases as in the first method. The integral is taken from a to b (whether $a < b$ or $a > b$), and numbers from a to b can be written in parametric form as $a + (b-a)u$ as u runs from 0 to 1. Therefore with the change of variables $t = a + (b-a)u$ the integral remainder equals

$$\begin{aligned} \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t) \, dt &= \int_0^1 \frac{((b-a)(1-u))^n}{n!} f^{(n+1)}(a + (b-a)u)(b-a) \, du \\ &= \frac{(b-a)^{n+1}}{n!} \int_0^1 (1-u)^n f^{(n+1)}(a + (b-a)u) \, du, \end{aligned}$$

so if $|f^{(n+1)}(t)| \leq M$ for all t from a to b then the absolute value of the integral remainder is at most

$$\begin{aligned} \frac{|b-a|^{n+1}}{n!} \int_0^1 (1-u)^n M \, du &= M \frac{|b-a|^{n+1}}{n!} \int_0^1 (1-u)^n \, du \\ &= M \frac{|b-a|^{n+1}}{n!} \frac{1}{n+1} \\ &= M \frac{|b-a|^{n+1}}{(n+1)!}. \end{aligned}$$

Remark 3.1. Taylor's inequality is not due to Taylor. In fact, Taylor's treatment of power series (in his book *Methodus Incrementorum Directa et Inversa*, written in 1715) was not concerned with justifications of convergence or error estimates, and preceded by almost 80 years the work of Lagrange and by over 100 years the work of Cauchy.