THE MEAN VALUE THEOREM AND INTEGRAL POWERS

KEITH CONRAD

As an illustration of many uses of the Mean Value Theorem, consider the following result, which was question A6 on the 32nd Putnam Exam [2] in 1971.[1] According to [2, p. 175], nobody answered it with a score of 8, 9, or 10 (the top score).

Theorem 1. *If every number* $2^t, 3^t, 4^t, 5^t, \ldots$ *(that is, every* n^t *where* $n = 2, 3, 4, 5, \ldots$*) is an integer, then* t *is* 0 *or a positive integer.*

Proof. Since 2^t is an integer, we must have $t \geq 0$. To prove t is an integer, we that it is impossible for t to be a non-integer: first $0 < t < 1$ can't happen, then $1 < t < 2$ can't happen, then $2 < t < 3$ can't happen, and so on, implying that t can only be one of the numbers $0, 1, 2, 3, \ldots$. We'll go through the argument for small $0 < t < 3$ first so the reader can see what is happening there before we treat the general case.

Case 1: We can't have $0 < t < 1$.

Assume $0 < t < 1$. For each integer $n \geq 1$, $(n+1)^t - n^t$ is an integer since n^t and $(n+1)^t$ are integers. The Mean Value Theorem implies

$$(n+1)^t - n^t = tx_n^{t-1}$$

for some x_n where $n < x_n < n+1$. Since $t - 1 < 0$, we have $x_n^{t-1} < n^{t-1}$, so

$$0 < (n+1)^t - n^t < tn^{t-1}.$$

Since $t - 1 < 0$, $tn^{t-1} \to 0$ as $n \to \infty$. Thus for large n, $0 < (n+1)^t - n^t < 1$, which is a contradiction: there is no integer between 0 and 1.

Case 2: We can't have $1 < t < 2$.

Assume $1 < t < 2$. For each integer $n \geq 1$, the number

$$((n+2)^t - (n+1)^t) - ((n+1)^t - n^t) = (n+2)^t - 2(n+1)^t + n^t$$

is an integer since n^t, $(n+1)^t$, and $(n+2)^t$ are all integers. By the Mean Value Theorem,

$$(n+1)^t - n^t = tx_n^{t-1} \quad \text{and} \quad (n+2)^t - (n+1)^t = tx_{n+1}^{t-1}$$

where $n < x_n < n+1$ and $n+1 < x_{n+1} < n+2$, so $x_n < x_{n+1}$. Then

$$\begin{aligned}(n+2)^t - 2(n+1)^t + n^t &= tx_{n+1}^{t-1} - tx_n^{t-1} \\ &= t(x_{n+1}^{t-1} - x_n^{t-1}) \\ &= t(t-1)y_n^{t-2}\end{aligned}$$

by the Mean Value Theorem, where $x_n < y_n < x_{n+1}$. Since $t > 1$, the numbers t and $t - 1$ are positive, so $t(t-1)y_n^{t-2} > 0$. Thus $(n+2)^t - 2(n+1)^t + n^t > 0$.

Since $n < x_n < y_n$ and $t - 2 < 0$,

$$0 < (n+2)^t - 2(n+1)^t + n^t = t(t-1)y_n^{t-2} < t(t-1)n^{t-2}.$$

[1] I first heard this result from a Mathoverflow question: https://mathoverflow.net/questions/ 17560.

As $n \to \infty$, $t(t-1)n^{t-2} \to 0$ due to the exponent $t-2$ being negative . Thus for large n,

$$0 < (n+2)^t - 2(n+1)^t + n^t < 1.$$

and that is contradiction since no integer is between 0 and 1.

Case 3: We can't have $2 < t < 3$.

Assume $2 < t < 3$. For each integer $n \geq 1$, we will look at the numbers

$$((n+3)^t - (n+2)^t) - 2((n+2)^t - (n+1)^t) + ((n+1)^t - n^t) = (n+3)^t - 3(n+2)^t + 3(n+1)^t - n^t,$$

which are integers since n^t, $(n+1)^t$, $(n+2)^t$, and $(n+3)^t$ are integers. We'll apply the Mean Value Theorem to each difference on the left side:

$$(n+1)^t - n^t = tx_n^{t-1} \quad (n+2)^t - (n+1)^t = tx_{n+1}^{t-1}, \quad \text{and} \quad (n+3)^t - (n+2)^t = tx_{n+2}^{t-1}$$

where $n < x_n < n+1$, $n+1 < x_{n+1} < n+2$, and $n+2 < x_{n+2} < n+3$. Then

$$\begin{aligned}(n+3)^t - 3(n+2)^t + 3(n+1)^t - n^t &= tx_{n+2}^{t-1} - 2tx_{n+1}^{t-1} + tx_n^{t-1}, \\ &= t((x_{n+2}^{t-1} - x_{n+1}^{t-1}) - (x_{n+1}^{t-1} - x_n^{t-1})) \\ &= t((t-1)y_{n+1}^{t-2} - (t-1)y_n^{t-2}) \\ &= t(t-1)(y_{n+1}^{t-2} - y_n^{t-2})\end{aligned}$$

by the Mean Value Theorem, where $x_n < y_n < x_{n+1}$ and $x_{n+1} < y_{n+1} < x_{n+2}$. Then $y_n < y_{n+1}$, so

$$y_{n+1}^{t-2} - y_n^{t-2} = (t-2)z_n^{t-3}$$

by the Mean Value Theorem, where $y_n < z_n < y_{n+1}$. Thus

$$(n+3)^t - 3(n+2)^t + 3(n+1)^t - n^t = t(t-1)(t-2)z_n^{t-3},$$

which is positive since $z_n > 0$ and $t > 2$. Since $n < x_n < y_n < z_n$ and $t-3 < 0$, $z_n^{t-3} < n^{t-3}$. Thus

$$0 < (n+3)^t - 3(n+2)^t + 3(n+1)^t - n^t = t(t-1)(t-2)z_n^{t-3} < t(t-1)(t-2)n^{t-3}.$$

We have $t(t-1)(t-2)n^{t-3} \to 0$ as $n \to \infty$ since $t-3 < 0$, so for large n,

$$0 < (n+3)^t - 3(n+2)^t - 3(n+1)^t + n^t < 1.$$

This is a contradiction since no integer is between 0 and 1.

General case: Hopefully the reader sees a general pattern emerging in order to prove that $k-1 < t < k$ is impossible for each positive integer k, which would mean that t can only be one of the numbers $0, 1, 2, 3, \ldots$, or in other words t must be 0 or a positive integer.

The strategy we use is based on the following identity that is inspired by the earlier cases: for positive integers k and n and all real numbers t,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (n+j)^t = t(t-1)\cdots(t-k+1)c^{t-k} \tag{1}$$

for some c in $(n, n+k)$ that depends on t, k, and n. We will actually prove a stronger result: for each positive integer k, real number t, and all $0 < a_0 < \cdots < a_k$,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} a_j^t = t(t-1)\cdots(t-k+1)c^{t-k} \tag{2}$$

for some c in (a_0, a_k) that depends on t, k, and the a_j's. (Equation (1) is the special case $a_j = n + j$.) To prove (2), we will use induction on k. The base case ($k = 1$) says $a_1^t - a_0^t = tc^{t-1}$ for some c in (a_0, a_1), which is the Mean Value Theorem for x^t on $[a_0, a_1]$.

To prove the inductive step, let $k \geq 2$ on the left side of (2) and split off the first and last terms on the the left side of (2). Then use the identity $\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}$ when $1 \leq j \leq k-1$ to show after recombining terms that

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_j^t = \sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}(a_{i+1}^t - a_i^t).$$

On the right side here, $a_{i+1}^t - a_i^t = tb_i^{t-1}$ for some b_i in (a_i, a_{i+1}) by the Mean Value Theorem, so $0 < b_0 < b_1 < \cdots < b_{k-1}$. Then

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_j^t = t\sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}b_i^{t-1}. \tag{3}$$

The sum on the right side of (3) is exactly what we can use in an inductive step: if (2) holds with $k-1$ in place of k (and all t's and a_j's), then by using $t-1$ in place of t and the b_i's in place of the a_j's, we can rewrite the sum on the right side of (3):

$$\begin{aligned}\sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}b_i^{t-1} &= (t-1)\cdots((t-1)-(k-1)+1)c^{(t-1)-(k-1)} \\ &= (t-1)\cdots(t-k+1)c^{t-k}\end{aligned}$$

for some c in (b_0, b_{k-1}). Thus

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_j^t = t(t-1)\cdots(t-k+1)c^{t-k}$$

where $a_0 < b_0 < c < b_{k-1} < a_k$, so c is in (a_0, a_k). That completes the proof of (2) for all k.

Now assume t is a real number such that $2^t, 3^t, 4^t, 5^t, \ldots$ are all integers. That 2^t is an integer implies $t \geq 0$. We will show for each positive integer k that we can't have $k-1 < t < k$, so $t = 0$ or t is a positive integer.

Assume $k-1 < t < k$, where k is a positive integer. Use (1):

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(n+j)^t = t(t-1)\cdots(t-k+1)c^{t-k},$$

where n is a positive integer and c is in $(n, n+k)$. The left side is an integer and all factors on the right side are positive since $k-1 < t$, so the left side is a positive integer. The exponent on c is negative since $t < k$. Since $n < c$ and $t - k < 0$, we have $c^{t-k} < n^{t-k}$, so

$$0 < \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(n+j)^t < t(t-1)\cdots(t-k+1)n^{t-k}.$$

As $n \to \infty$, we have $n^{t-k} \to 0$, so for large enough n

$$0 < \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (n+j)^t < 1,$$

and that is a contradiction since the number between 0 and 1 is an integer. So for each positive integer k we can't have $k - 1 < t < k$. □

Remark 2. Using the six exponentials theorem from transcendental number theory, it can be shown that if we assume only that 2^t, 3^t, and 5^t are integers, then t is 0 or a positive integer. (This is also true when 2, 3, and 5 are replaced by any three multiplicatively independent positive integers.)

It is expected that if we assume only that 2^t and 3^t are integers then t has to be 0 or a positive integer, but that remains an unsolved problem. (The same conclusion should also hold when 2 and 3 are replaced by any two multiplicatively independent positive integers, and the case of two different primes was posed by Alaoglu and Erdős [1, p. 449].) This would be a consequence of the unproved four exponentials conjecture, which in turn is a consequence of Schanuel's conjecture.

References

[1] L. Alaoglu and P. Erdős, "On Highly Composite and Similar Numbers," *Trans. Amer. Math. Soc.* **56** (1944), 448–469.

[2] J. H. McKay, The William Lowell Putnam Mathematical Competition, Amer. Math. Monthly **80** (1973), pp. 172–175. URL `https://www.jstor.org/stable/2318375?seq=6`