THE MEAN VALUE THEOREM AND INTEGRAL POWERS

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As an illustration of many uses of the Mean Value Theorem, consider the following result, which was question A6 on the 32nd Putnam Exam [2] in 1971.[1] According to [2, p. 175], nobody answered it with a score of 8, 9, or 10 (the top score).

Theorem 1. *If every number* $2^t, 3^t, 4^t, 5^t, \ldots$ (*that is, every* n^t *where* $n = 2, 3, 4, 5, \ldots$) *is an integer, then* t *is* 0 *or a positive integer.*

Proof. Since 2^t is an integer, we must have $t \geq 0$. To prove t is an integer, we prove that it is impossible for t to be a non-integer: first $0 < t < 1$ can't happen, then $1 < t < 2$ can't happen, then $2 < t < 3$ can't happen, and so on, implying that t can only be one of the numbers $0, 1, 2, 3, \ldots$. We'll go through the argument for small $0 < t < 3$ first so the reader can see what is happening there before we treat the general case.

Case 1: We can't have $0 < t < 1$.

Assume $0 < t < 1$. For each integer $n \geq 1$, $(n+1)^t - n^t$ is an integer since n^t and $(n+1)^t$ are integers. The Mean Value Theorem implies

$$(n+1)^t - n^t = t x_n^{t-1}$$

for some x_n where $n < x_n < n+1$. Since $t - 1 < 0$, we have $x_n^{t-1} < n^{t-1}$, so

$$0 < (n+1)^t - n^t < t n^{t-1}.$$

Since $t - 1 < 0$, $tn^{t-1} \to 0$ as $n \to \infty$. Thus for large n, $0 < (n+1)^t - n^t < 1$, which is a contradiction: there is no integer between 0 and 1.

Case 2: We can't have $1 < t < 2$.

Assume $1 < t < 2$. For each integer $n \geq 1$, the number

$$((n+2)^t - (n+1)^t) - ((n+1)^t - n^t) = (n+2)^t - 2(n+1)^t + n^t$$

is an integer since n^t, $(n+1)^t$, and $(n+2)^t$ are all integers. By the Mean Value Theorem,

$$(n+1)^t - n^t = t x_n^{t-1} \quad \text{and} \quad (n+2)^t - (n+1)^t = t x_{n+1}^{t-1}$$

where $n < x_n < n+1$ and $n+1 < x_{n+1} < n+2$, so $x_n < x_{n+1}$. Then

$$\begin{aligned} (n+2)^t - 2(n+1)^t + n^t &= t x_{n+1}^{t-1} - t x_n^{t-1} \\ &= t(x_{n+1}^{t-1} - x_n^{t-1}) \\ &= t(t-1) y_n^{t-2} \end{aligned}$$

by the Mean Value Theorem, where $x_n < y_n < x_{n+1}$. Since $t > 1$, the numbers t and $t - 1$ are positive, so $t(t-1)y_n^{t-2} > 0$. Thus $(n+2)^t - 2(n+1)^t + n^t > 0$.

Since $n < x_n < y_n$ and $t - 2 < 0$,

$$0 < (n+2)^t - 2(n+1)^t + n^t = t(t-1)y_n^{t-2} < t(t-1)n^{t-2}.$$

[1] I first heard this result from a Mathoverflow question: https://mathoverflow.net/questions/17560.

As $n \to \infty$, $t(t-1)n^{t-2} \to 0$ due to the exponent $t-2$ being negative . Thus for large n,

$$0 < (n+2)^t - 2(n+1)^t + n^t < 1.$$

and that is contradiction since no integer is between 0 and 1.

Case 3: We can't have $2 < t < 3$.

Assume $2 < t < 3$. For each integer $n \geq 1$, we will look at the numbers

$$((n+3)^t - (n+2)^t) - 2((n+2)^t - (n+1)^t) + ((n+1)^t - n^t) = (n+3)^t - 3(n+2)^t + 3(n+1)^t - n^t,$$

which are integers since n^t, $(n+1)^t$, $(n+2)^t$, and $(n+3)^t$ are integers. We'll apply the Mean Value Theorem to each difference on the left side:

$$(n+1)^t - n^t = tx_n^{t-1} \quad (n+2)^t - (n+1)^t = tx_{n+1}^{t-1}, \quad \text{and} \quad (n+3)^t - (n+2)^t = tx_{n+2}^{t-1}$$

where $n < x_n < n+1$, $n+1 < x_{n+1} < n+2$, and $n+2 < x_{n+2} < n+3$. Then

$$\begin{aligned}(n+3)^t - 3(n+2)^t + 3(n+1)^t - n^t &= tx_{n+2}^{t-1} - 2tx_{n+1}^{t-1} + tx_n^{t-1}, \\ &= t((x_{n+2}^{t-1} - x_{n+1}^{t-1}) - (x_{n+1}^{t-1} - x_n^{t-1})) \\ &= t((t-1)y_{n+1}^{t-2} - (t-1)y_n^{t-2}) \\ &= t(t-1)(y_{n+1}^{t-2} - y_n^{t-2})\end{aligned}$$

by the Mean Value Theorem, where $x_n < y_n < x_{n+1}$ and $x_{n+1} < y_{n+1} < x_{n+2}$. Then $y_n < y_{n+1}$, so

$$y_{n+1}^{t-2} - y_n^{t-2} = (t-2)z_n^{t-3}$$

by the Mean Value Theorem, where $y_n < z_n < y_{n+1}$. Thus

$$(n+3)^t - 3(n+2)^t + 3(n+1)^t - n^t = t(t-1)(t-2)z_n^{t-3},$$

which is positive since $z_n > 0$ and $t > 2$. Since $n < x_n < y_n < z_n$ and $t-3 < 0$, $z_n^{t-3} < n^{t-3}$. Thus

$$0 < (n+3)^t - 3(n+2)^t + 3(n+1)^t - n^t = t(t-1)(t-2)z_n^{t-3} < t(t-1)(t-2)n^{t-3}.$$

We have $t(t-1)(t-2)n^{t-3} \to 0$ as $n \to \infty$ since $t-3 < 0$, so for large n,

$$0 < (n+3)^t - 3(n+2)^t - 3(n+1)^t + n^t < 1.$$

This is a contradiction since no integer is between 0 and 1.

General case: Hopefully the reader sees a general pattern emerging in order to prove that $k-1 < t < k$ is impossible for each positive integer k, which would mean that t can only be one of the numbers $0, 1, 2, 3, \ldots$, or in other words t must be 0 or a positive integer.

The strategy we use is based on the following identity that is inspired by the earlier cases: for positive integers k and n and all real numbers t,

$$\text{(1)} \qquad \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (n+j)^t = t(t-1)\cdots(t-k+1)c^{t-k}$$

for some c in $(n, n+k)$ that depends on t, k, and n. We will actually prove a stronger result: for each positive integer k, real number t, and all $0 < a_0 < \cdots < a_k$,

$$\text{(2)} \qquad \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} a_j^t = t(t-1)\cdots(t-k+1)c^{t-k}$$

for some c in (a_0, a_k) that depends on t, k, and the a_j's. (Equation (1) is the special case $a_j = n + j$.) To prove (2), we will use induction on k. The base case ($k = 1$) says $a_1^t - a_0^t = tc^{t-1}$ for some c in (a_0, a_1), which is the Mean Value Theorem for x^t on $[a_0, a_1]$.

To prove the inductive step, let $k \geq 2$ on the left side of (2) and split off the first and last terms on the the left side of (2). Then use the identity $\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}$ when $1 \leq j \leq k - 1$ to show after recombining terms that

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_j^t = \sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}(a_{i+1}^t - a_i^t).$$

On the right side here, $a_{i+1}^t - a_i^t = tb_i^{t-1}$ for some b_i in (a_i, a_{i+1}) by the Mean Value Theorem, so $0 < b_0 < b_1 < \cdots < b_{k-1}$. Then

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_j^t = t\sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}b_i^{t-1}. \tag{3}$$

The sum on the right side of (3) is exactly what we can use in an inductive step: if (2) holds with $k - 1$ in place of k (and all t's and a_j's), then by using $t - 1$ in place of t and the b_i's in place of the a_j's, we can rewrite the sum on the right side of (3):

$$\begin{aligned}\sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}b_i^{t-1} &= (t-1)\cdots((t-1)-(k-1)+1)c^{(t-1)-(k-1)} \\ &= (t-1)\cdots(t-k+1)c^{t-k}\end{aligned}$$

for some c in (b_0, b_{k-1}). Thus

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_j^t = t(t-1)\cdots(t-k+1)c^{t-k}$$

where $a_0 < b_0 < c < b_{k-1} < a_k$, so c is in (a_0, a_k). That completes the proof of (2) for all k.

Now assume t is a real number such that $2^t, 3^t, 4^t, 5^t, \ldots$ are all integers. That 2^t is an integer implies $t \geq 0$. We will show for each positive integer k that we can't have $k - 1 < t < k$, so $t = 0$ or t is a positive integer.

Assume $k - 1 < t < k$, where k is a positive integer. Use (1):

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(n+j)^t = t(t-1)\cdots(t-k+1)c^{t-k},$$

where n is a positive integer and c is in $(n, n+k)$. The left side is an integer and all factors on the right side are positive since $k - 1 < t$, so the left side is a positive integer. The exponent on c is negative since $t < k$. Since $n < c$ and $t - k < 0$, we have $c^{t-k} < n^{t-k}$, so

$$0 < \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(n+j)^t < t(t-1)\cdots(t-k+1)n^{t-k}.$$

As $n \to \infty$, we have $n^{t-k} \to 0$, so for large enough n

$$0 < \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (n+j)^t < 1,$$

and that is a contradiction since the number between 0 and 1 is an integer. So for each positive integer k we can't have $k - 1 < t < k$. □

Remark 2. Using the six exponentials theorem from transcendental number theory, it can be shown that if we assume only that 2^t, 3^t, and 5^t are integers, then t is 0 or a positive integer. (This is also true when 2, 3, and 5 are replaced by any three multiplicatively independent positive integers.)

It is expected that if we assume only that 2^t and 3^t are integers then t has to be 0 or a positive integer, but that remains an unsolved problem. (The same conclusion should also hold when 2 and 3 are replaced by any two multiplicatively independent positive integers, and the case of two different primes was posed by Alaoglu and Erdős [1, p. 449].) This would be a consequence of the unproved four exponentials conjecture, which in turn is a consequence of Schanuel's conjecture.

References

[1] L. Alaoglu and P. Erdős, "On Highly Composite and Similar Numbers," *Trans. Amer. Math. Soc.* **56** (1944), 448–469.

[2] J. H. McKay, The William Lowell Putnam Mathematical Competition, Amer. Math. Monthly **80** (1973), pp. 172–175. URL `https://www.jstor.org/stable/2318375?seq=6`