

A q -ANALOGUE OF MAHLER EXPANSIONS I

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ABSTRACT. We examine a q -analogue of Mahler expansions for continuous functions in p -adic analysis, replacing binomial coefficient polynomials $\binom{x}{n}$ with a q -analogue $\binom{x}{n}_q$ for a p -adic variable q with $|q - 1|_p < 1$. Mahler expansions are recovered at $q = 1$ and we consider the p -adic q -Gamma function $\Gamma_{p,q}$ of Koblitz relative to its q -Mahler expansion.

1. INTRODUCTION

Let \mathbf{Z}_p be the p -adic integers, \mathbf{Q}_p the p -adic rationals, and K a field extension of \mathbf{Q}_p which is complete with respect to a nonarchimedean absolute value $|\cdot|_p$, normalized by $|p|_p = 1/p$.

About forty years ago, Mahler introduced in [18] an expansion for continuous functions from \mathbf{Z}_p to K using special polynomials. Specifically, he observed that the n th binomial coefficient polynomial

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

sends \mathbf{Z}_p to \mathbf{Z}_p (it sends \mathbf{Z} to $\mathbf{Z} \subset \mathbf{Z}_p$, then use continuity), so $|\binom{x}{n}|_p \leq 1$ for all $x \in \mathbf{Z}_p$. Therefore for any sequence $c_n \in K$ with $\lim_{n \rightarrow \infty} c_n = 0$, the series

$$f(x) = \sum_{n \geq 0} c_n \binom{x}{n}$$

defines a continuous function $\mathbf{Z}_p \rightarrow K$. Mahler proved every continuous function from \mathbf{Z}_p to K arises uniquely in this way, with

$$c_n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(k), \quad \sup_{x \in \mathbf{Z}_p} |f(x)|_p = \max_{n \geq 0} |c_n|_p.$$

The c_n are called the *Mahler coefficients* of f and the series $\sum c_n \binom{x}{n}$ is called the *Mahler expansion* of f .

In this paper a q -analogue of the Mahler expansion is studied, where q is a p -adic variable.

To set up the framework for our ideas, first we recall the philosophy of q -analogues over \mathbf{R} and \mathbf{C} . For a complex number q other than 1, define the q -analogue of a positive integer n to be

$$(n)_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

As $q \rightarrow 1$, $(n)_q \rightarrow n$, and this is the hallmark of a q -analogue: the limit as $q \rightarrow 1$ recovers the classical object. There are q -analogues of most functions in classical analysis [9]. For

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example, the geometric series

$$(1 - z)^{-a} = \sum_{n \geq 0} \frac{a(a+1) \cdots (a+n-1)}{n!} z^n$$

for $|z| < 1$ and $a \in \mathbf{C}$ has the q -analogue

$$1 + \frac{q^a - 1}{q - 1} z + \frac{(q^a - 1)(q^{a+1} - 1)}{(q - 1)(q^2 - 1)} z^2 + \cdots = \prod_{n \geq 0} \frac{1 - q^{a+n} z}{1 - q^n z},$$

where the infinite product converges for $|q| < 1$. The analytic treatment of q -series in \mathbf{C} usually assumes $|q| < 1$ or $0 < q < 1$. However, many results make sense in a formal way, allowing q to be viewed as an indeterminate. The study of q -analogues has connections with a number of areas of mathematics, such as partitions, modular functions, and quantum groups.

The Mahler expansion in p -adic analysis uses binomial coefficient polynomials $\binom{x}{n}$, $x \in \mathbf{Z}_p$. For $q \in K$ with $|q - 1|_p < 1$ (the p -adic substitute for the condition $|q| < 1$ in \mathbf{C}), we will use q -analogues $\binom{x}{n}_q$. These are exponential functions of $x \in \mathbf{Z}_p$ if q is not a root of unity, and are locally polynomials in x if q is a root of unity. In particular, $\binom{x}{n}_1 = \binom{x}{n}$. The q -analogue of Mahler's theorem is

THEOREM. *For a complete extension field K/\mathbf{Q}_p and $q \in K$ with $|q - 1|_p < 1$, every continuous function $f: \mathbf{Z}_p \rightarrow K$ has a unique expansion*

$$f(x) = \sum_{n \geq 0} c_{n,q} \binom{x}{n}_q$$

where $c_{n,q} \in K$ and $c_{n,q} \rightarrow 0$ as $n \rightarrow \infty$. Furthermore,

$$c_{n,q} = \sum_{k=0}^n \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} f(k), \quad \sup_{x \in \mathbf{Z}_p} |f(x)|_p = \max_{n \geq 0} |c_{n,q}|_p.$$

About twenty years ago, van Hamme [23] proved the p -adic analogue of a result of F. H. Jackson on real q -series, thereby giving explicit polynomial approximations for continuous functions on certain compact-open subsets V_q of \mathbf{Z}_p . The subset and the approximating polynomials depend on a parameter $q \in \mathbf{Z}_p^\times$ which can not be a root of unity. A. Verdoort has continued this work. The point of view of van Hamme and Verdoort is largely compatible with the one presented in Section 3 after a change of variables, although our approach, unlike theirs, permits a passage to the limit as $q \rightarrow 1$ to recover Mahler's theorem at $q = 1$.

The structure of the paper is as follows. In Section 2 we review some properties of q -analogues, where q will be treated mostly as an indeterminate. In Section 3 we let q be a p -adic variable and discuss the q -analogue of Mahler's theorem. Four proofs are given, having individual advantages. Because this paper may be of interest to people who work in p -adic analysis but not in q -series, and vice versa, we give extra details in Sections 2 and 3 for results that are well-known to those familiar with one of these areas but not the other. In Section 4 we discuss properties of q -Mahler expansions. One aspect which is not apparent in the classical case $q = 1$ is the role of the p -adic logarithm in classifying differentiability in terms of q -Mahler expansions. In Section 5 we discuss the q -Mahler expansion of the p -adic q -Gamma function of Koblitz.

Here is a brief list of notation.

\mathbf{N} is the set of natural numbers $\{0, 1, 2, \dots\}$.

\mathbf{Z}_p is the ring of p -adic integers.

\mathbf{Q}_p is the field of p -adic numbers.

ζ denotes a root of unity.

Φ_n is the n th cyclotomic polynomial.

For a function f on \mathbf{Z}_p , $(E^y f)(x) = f(x + y)$ is the shift by y . In particular, $(Ef)(x) = f(x + 1)$.

Let $(K, |\cdot|)$ be a complete extension field of \mathbf{Q}_p with $|p| = 1/p$. The continuous functions from \mathbf{Z}_p to K will be denoted $C(\mathbf{Z}_p, K)$ and topologized by the sup-norm $|f|_{\text{sup}} := \sup_{x \in \mathbf{Z}_p} |f(x)|$. (We only consider p -adic absolute values, so we write $|\cdot|$ rather than $|\cdot|_p$.)

A function $\mathbf{Z}_p \rightarrow K$ is called *analytic* if it is given by a single power series that converges on \mathbf{Z}_p . It is called *locally analytic* if it is locally expressible by a power series around each point of \mathbf{Z}_p .

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2. A REVIEW OF q -FORMALISM

Here we recall the features of q -analogues that are needed for our purposes, generally insofar as q can be treated as an indeterminate. Some remarks will be made about specializing q , especially at roots of unity. The focus will be on properties of q -binomial coefficients and q -difference operators.

For an integer n and an indeterminate q , the q -analogue of n is

$$(n)_q := \frac{q^n - 1}{q - 1}.$$

For example, $(0)_q = 0$, $(1)_q = 1$, $(2)_q = 1 + q$, $(-1)_q = -1/q$.

When $n \geq 1$, $(n)_q = 1 + q + \dots + q^{n-1}$ is a polynomial in $\mathbf{Z}[q]$.

For any integers m and n ,

$$(2.1) \quad (-n)_q = -\frac{1}{q^n}(n)_q, \quad (n)_{1/q} = \frac{1}{q^{n-1}}(n)_q, \quad (mn)_q = (m)_q(n)_{q^m}.$$

Specializing $q = 1$, $(n)_q$ becomes n .

The q -factorials are

$$(n)_q! := \begin{cases} 1, & n = 0; \\ (n)_q(n-1)_q \cdots (1)_q, & n \geq 1. \end{cases}$$

For example, $(1)_q! = 1$, $(2)_q! = 1 + q$, $(3)_q! = 1 + 2q + 2q^2 + q^3$, and

$$(2.2) \quad (n)_{1/q}! = \frac{1}{q^{n(n-1)/2}}(n)_q!.$$

The q -binomial coefficient for nonnegative integers m and n with $m \geq n$ is

$$\begin{aligned} \binom{m}{n}_q &:= \frac{(m)_q!}{(n)_q!(m-n)_q!} \\ &= \frac{(m)_q(m-1)_q \cdots (m-n+1)_q}{(n)_q!} \\ &= \frac{(q^m-1)(q^{m-1}-1) \cdots (q^{m-n+1}-1)}{(q^n-1)(q^{n-1}-1) \cdots (q-1)}. \end{aligned}$$

We use the second or third expression to extend the definition of $\binom{m}{n}_q$ to any integer m . These functions go back to Gauss [10, p. 16], so they are also called Gaussian coefficients.

The first few q -binomial coefficients are

$$\binom{m}{0}_q = 1, \quad \binom{m}{1}_q = (m)_q = \frac{q^m-1}{q-1}, \quad \binom{m}{2}_q = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)}.$$

For $m \geq n$, $\binom{m}{n}_q = \binom{m}{m-n}_q$, and (as a rational function in q) $\binom{m}{n}_q = 0$ precisely when $0 \leq m < n$. The q -binomial coefficient may vanish in other cases numerically, e.g., $\binom{4}{2}_q = (1+q^2)(1+q+q^2)$, so $\binom{4}{2}_i = 0$.

The following result is essentially due to Gauss [10, p. 17].

THEOREM 2.1. *For fixed integers $m \geq n \geq 0$, $\binom{m}{n}_q \in \mathbf{Z}[q]$ with degree $n(m-n)$.*

Proof. The degree follows from the definition, once we know $\binom{m}{n}_q$ is a polynomial in q .

We give Gauss' proof that $\binom{m}{n}_q \in \mathbf{Z}[q]$ and then an alternate proof that seems to be new.

The Pascal's triangle recursion for binomial coefficients generalizes (for all m in \mathbf{Z}) to

$$(2.3) \quad \binom{m}{n}_q = \binom{m-1}{n-1}_q + q^n \binom{m-1}{n}_q = q^{m-n} \binom{m-1}{n-1}_q + \binom{m-1}{n}_q$$

(when $m \geq n$, replace n by $m-n$ to obtain either recursion from the other), and iterating the second recursion gives

$$\binom{m+n+1}{n+1}_q = q^m \binom{m+n}{n}_q + \binom{m+n}{n+1}_q = \sum_{k=0}^m q^k \binom{k+n}{n}_q.$$

So $\binom{m}{n} \in \mathbf{Z}[q]$ by induction on n (and actually all the coefficients are nonnegative).

As an alternate proof, the irreducible factors of the rational function $\binom{m}{n}_q$ are cyclotomic polynomials. The multiplicity of the j th cyclotomic polynomial $\Phi_j(q)$ as a factor of $(n)_q!$ is $[n/j]$, so its multiplicity as a factor of $\binom{m}{n}_q$ is $[m/j] - [n/j] - [(m-n)/j]$, which is 0 or 1. This shows for $m \geq n$ not only that $\binom{m}{n}_q$ is a polynomial in q , but that its irreducible factors are all simple factors and $\Phi_j(q)$ is a factor precisely when the units' digit of m in base j is less than the units' digit of n in base j . I thank Ira Gessel for a simplification to the original form of this alternate proof. \square

Further identities for all $m \in \mathbf{Z}$ (and $k \geq j \geq 0$) are

$$(2.4) \quad \binom{m}{n}_q = \frac{(m)_q}{(n)_q} \binom{m-1}{n-1}_q, \quad \binom{m}{n}_{1/q} = \frac{1}{q^{n(m-n)}} \binom{m}{n}_q, \quad \binom{m}{k}_q \binom{k}{j}_q = \binom{m}{j}_q \binom{m-j}{k-j}_q,$$

$$(2.5) \quad \binom{-m}{n}_q = (-1)^n q^{-n(n-1)/2 - mn} \binom{m+n-1}{n}_q = (-1)^n q^{-n(n+1)/2} \binom{m+n-1}{n}_{1/q}.$$

For example, $\binom{-1}{n}_q = (-1)^n q^{-n(n+1)/2}$. By (2.5), for $m > 0$ $\binom{-m}{n}_q$ is a polynomial in $1/q$ with degree $n(n-1)/2 + mn$ whose coefficients are nonzero integers with sign $(-1)^n$.

The next result is a q -analogue of the binomial theorem, the q -binomial theorem. It goes back to Cauchy [4, p. 46, Eq. 18].

THEOREM 2.2. For $m \geq 1$,

$$(1+T)(1+qT) \cdots (1+q^{m-1}T) = \prod_{i=0}^{m-1} (1+q^i T) = \sum_{k=0}^m \binom{m}{k}_q q^{k(k-1)/2} T^k.$$

Equivalently, for commuting variables X and Y ,

$$(X+Y)(X+qY) \cdots (X+q^{m-1}Y) = \prod_{i=0}^{m-1} (X+q^i Y) = \sum_{k=0}^m \binom{m}{k}_q q^{k(k-1)/2} X^{m-k} Y^k.$$

Proof. Following Cauchy [4, p. 51], let $h(T) = \prod_{i=0}^{m-1} (1+q^i T) = \sum_{k=0}^m a_k T^k$. Then $(1+T)h(qT) = h(T)(1+q^m T)$. Equating coefficients of equal powers of T ,

$$a_k = \frac{q^m - q^{k-1}}{q^k - 1} a_{k-1} = \frac{q^{m-k+1} - 1}{q^k - 1} q^{k-1} a_{k-1},$$

so $a_k = \binom{m}{k}_q q^{k(k-1)/2}$. □

In particular,

$$(2.6) \quad (X-1)(X-q) \cdots (X-q^{m-1}) = \sum_{k=0}^m \binom{m}{k}_q (-1)^k q^{k(k-1)/2} X^{m-k}.$$

Actually, the idea of replacing T by qT to express q -products as q -series goes back to Euler [7, Ch. XVI, §§306, 307].

The $q^{k(k-1)/2}$ term that arises in the q -binomial theorem can be removed from explicit appearance. Define the n th q -power of a polynomial $f(T)$ to be $f^{(0;q)} = 1$ and $f^{(n;q)} := f(T)f(qT) \cdots f(q^{n-1}T)$ for $n \geq 1$. Then the q -binomial theorem becomes

$$(1+T)^{(m;q)} = \sum_{k=0}^m \binom{m}{k}_q T^{(k;q)}.$$

We can consider q -deformed powers of a polynomial in several variables by singling out one variable, e.g., in two variables

$$f(X, Y)^{(n;q)} := f(X, Y)f(X, qY) \cdots f(X, q^{n-1}Y).$$

This will appear later in the case of $(X \pm Y)^{(n;q)}$, whose value at $X = x$, $Y = y$ will be written with abuse of notation as $(x \pm y)^{(n;q)}$. For example,

$$(x+0)^{(n;q)} = x^m, \quad (0+y)^{(n;q)} = q^{n(n-1)/2} y^n, \quad \binom{m}{k}_q = \frac{(q^m - 1)^{(k;q)}}{(q^k - 1)^{(k;q)}}.$$

The q -Vandermonde formula for $\binom{m_1+m_2}{k}_q$ is proven as for ordinary binomial coefficients.

THEOREM 2.3. For $m_1, m_2 \geq 0$, $\binom{m_1+m_2}{k}_q = \sum_{j=0}^k \binom{m_1}{j}_q \binom{m_2}{k-j}_q q^{j(m_2-(k-j))}$.

Note the asymmetric roles of j and $k - j$ in the exponent of q on the right side.

Proof. Compare the coefficient of T^k on both sides of

$$\prod_{i=0}^{m_1+m_2-1} (1 + q^i T) = \prod_{i=0}^{m_2-1} (1 + q^i T) \prod_{i=0}^{m_1-1} (1 + q^i q^{m_2} T).$$

□

By a specialization argument, Theorem 2.3 is true for all integers m_1, m_2 , possibly negative.

The following simple fact will be used when we let q vary p -adically.

THEOREM 2.4. For $m, n \geq 0$, $\binom{m}{n}_{q_1} - \binom{m}{n}_{q_2} \in (q_1 - q_2)\mathbf{Z}[q_1, q_2]$.

Proof. For all $i \geq 0$, $q_1^i - q_2^i \in (q_1 - q_2)\mathbf{Z}[q_1, q_2]$. □

We now discuss the value of $\binom{m}{n}_q$ for $m \geq n$ when q is specialized to various numbers.

When $q = 1$, $\binom{m}{n}_1 = \binom{m}{n}$ counts the number of n element subsets of an m element set. When q is a prime power, $\binom{m}{n}_q$ counts the number of n -dimensional subspaces of an m -dimensional vector space over the field of size q . This suggests the possibility of proving identities for q -binomial coefficients by letting q run through (infinitely many) prime powers and interpreting the identity as a combinatorial statement in linear algebra over finite fields. See [11] for this approach.

We now consider the case when q is specialized to a root of unity. For ζ a root of unity of order b and $n < b$, the value of $\binom{m}{n}_\zeta$ can be computed directly from the definition, since $(n)_\zeta! \neq 0$. The next theorem reduces the evaluation of all $\binom{m}{n}_\zeta$ to the case when $n < b$.

THEOREM 2.5. Let ζ be a root of unity of order b .

i) For integers k and l , with $l \geq 0$, $\binom{bk}{bl}_\zeta = \binom{k}{l}$.

ii) For integers k and l with $l \geq 0$ and $0 \leq r, s < b$, $\binom{bk+r}{bl+s}_\zeta = \binom{bk}{bl}_\zeta \binom{r}{s}_\zeta = \binom{k}{l} \binom{r}{s}_\zeta$.

In particular, if $n < b$ and $m_1 \equiv m_2 \pmod{b}$, then $\binom{m_1}{n}_\zeta = \binom{m_2}{n}_\zeta$.

Proof. i)

$$\binom{bk}{bl}_q = \prod_{j=0}^{bl-1} \frac{q^{bk-j} - 1}{q^{bl-j} - 1} = \prod_{\substack{j=0 \\ j \not\equiv 0 \pmod{b}}}^{bl-1} \frac{q^{bk-j} - 1}{q^{bl-j} - 1} \cdot \prod_{i=0}^{l-1} \frac{q^{b(k-i)} - 1}{q^{b(l-i)} - 1}.$$

At $q = \zeta$, the right side becomes $\prod_{i=0}^{l-1} (k-i)/(l-i) = \binom{k}{l}$.

ii) First we show $\binom{bk+a}{bl}_\zeta = \binom{bk+a-1}{bl}_\zeta$ when a is not divisible by b . Setting $m = bk + a$, $n = bl$, and $q = \zeta$ in the equation $\binom{m}{n}_q = \frac{(m)_q}{(m-n)_q} \binom{m-1}{n}_q$, we get what we want. So the theorem is true for $s = 0$. For $s \geq 1$,

$$\binom{bk+r}{bl+s}_q = \frac{(bk+r)_q (bk+r-1)_q \cdots (bk+r-s+1)_q}{(bl+s)_q (bl+s-1)_q \cdots (bl+1)_q} \binom{bk+r-s}{bl}_q.$$

None of the terms $(bl+j)_q$ appearing in the denominator vanishes at $q = \zeta$, so we can evaluate and find

$$\binom{bk+r}{bl+s}_\zeta = \frac{(r)_\zeta (r-1)_\zeta \cdots (r-s+1)_\zeta}{(s)_\zeta (s-1)_\zeta \cdots (1)_\zeta} \binom{bk+r-s}{bl}_\zeta = \binom{r}{s}_\zeta \binom{bk+r-s}{bl}_\zeta.$$

□

COROLLARY 2.1. Let ζ be a root of unity of order b and $n \in \mathbf{N}$. For m running through a fixed residue class mod b , $\binom{m}{n}_\zeta$ is a polynomial in m .

Proof. By Theorem 2.5 (ii), $\binom{m}{n}_\zeta$ is a polynomial in $[m/b] = (m - r)/b$ and r is fixed. \square

EXAMPLES.

$$\binom{19}{5}_{-1} = \binom{18+1}{4+1}_{-1} = \binom{9}{2} \binom{1}{1}_{-1} = 36,$$

$$\binom{17}{10}_i = \binom{16+1}{4+2}_i = \binom{4}{2} \binom{1}{2}_i = 0, \quad \binom{-5}{6}_i = \binom{-8+3}{4+2}_i = \binom{-2}{1} \binom{3}{2}_i = -2i.$$

The periodicity of $\binom{m}{n}_\zeta$ in $m \bmod b$, stated at the end of Theorem 2.5, can also be verified by computing $\binom{m+b}{n}_\zeta - \binom{m}{n}_\zeta$ with the q -Vandermonde formula.

Theorem 2.5 (and an extension to q -multinomial coefficients) can be proven by group actions [21].

For a root of unity ζ of order b , that $\binom{b}{n}_\zeta = 0$ for $1 \leq n \leq b-1$ can be seen without Theorem 2.5, since the numerator of $\binom{b}{n}_q$ vanishes at $q = \zeta$ while the denominator does not, or (using Theorem 2.2) since $\prod_{j=0}^{b-1} (1 + \zeta^j T) = 1 - (-T)^b$. Stated in terms of the b th cyclotomic polynomial $\Phi_b(q)$, this vanishing becomes

$$(2.7) \quad \binom{b}{n}_q \equiv 0 \pmod{\Phi_b(q)}$$

when $1 \leq n \leq b-1$, which is also clear from the second proof of Theorem 2.1. Specializing (2.7) at $q = 1$, we recover the familiar integer congruence $\binom{p^N}{n} \equiv 0 \pmod{p}$ when $b = p^N$ is a power of a prime p . Since

$$\Phi_{p^N}(q) = \frac{q^{p^N} - 1}{q^{p^{N-1}} - 1} = (p)_{q^{p^{N-1}}},$$

when $b = p^N$ (2.7) can be written as $\binom{p^N}{n}_q \equiv 0 \pmod{(p)_{q^{p^{N-1}}}}$.

The q -analogue of the exponential series was introduced by Jackson [13]:

$$E_q(X) := \sum_{n \geq 0} \frac{X^n}{(n)_q!}.$$

(In the literature, the notation $E_q(X)$ may denote a slightly different series.) Jackson's q -version of $e^{x+y} = e^x e^y$ comes from (2.2) and the q -binomial theorem:

$$(2.8) \quad E_q(X) E_{1/q}(Y) = \sum_{n \geq 0} \frac{(X+Y)(X+qY) \cdots (X+q^{n-1}Y)}{(n)_q!} = \sum_{n \geq 0} \frac{(X+Y)^{(n;q)}}{(n)_q!}.$$

In particular,

$$(2.9) \quad E_q(X)^{-1} = E_{1/q}(-X) = \sum_{n \geq 0} (-1)^n q^{n(n-1)/2} \frac{X^n}{(n)_q!}.$$

We now discuss q -difference operators. Powers Δ^n of the difference operator Δ , where $(\Delta h)(x) = h(x+1) - h(x)$ (here and in the rest of this section, x is an integer variable), play a role in Mahler expansions which will be taken over in the q -analogue by a sequence of operators Δ_q^n first introduced by Jackson [14, p. 256], [15, p. 145].

The powers of Δ behave nicely on binomial coefficients, namely

$$\Delta^m \binom{x}{n} = \begin{cases} \binom{x}{n-m}, & \text{if } m \leq n; \\ 0, & \text{if } m > n. \end{cases}$$

The q -analogue of powers of Δ arise naturally by considering differences of q -binomial coefficients.

First, note that in analogy with $\Delta \binom{x}{n} = \binom{x}{n-1}$,

$$\Delta \binom{x}{n}_q = q^{x+1-n} \binom{x}{n-1}_q.$$

Then, guided by the equation $\Delta^2 \binom{x}{n} = \Delta \binom{x+1}{n} - \Delta \binom{x}{n} = \binom{x}{n-2}$, we compute

$$\Delta \binom{x+1}{n}_q = q^{x+2-n} \binom{x+1}{n-1}_q,$$

so we're naturally led to calculate not $\Delta \binom{x+1}{n}_q - \Delta \binom{x}{n}_q$ but

$$\Delta \binom{x+1}{n}_q - q \Delta \binom{x}{n}_q = q^{x+2-n} \left(\binom{x+1}{n-1}_q - \binom{x}{n-1}_q \right) = q^{2(x+2-n)} \binom{x}{n-2}_q.$$

Let $(Eh)(x) = h(x+1)$ be the shift operator, so we've computed

$$(E-I) \binom{x}{n}_q = q^{x+1-n} \binom{x}{n-1}_q, \quad (E-I)(E-q) \binom{x}{n}_q = q^{2(x+2-n)} \binom{x}{n-2}_q.$$

Of course $n \geq 1$ and $n \geq 2$ for these respective equations.

Experience with q -deformed products as in the q -binomial theorem now makes the following definition natural: $\Delta_q^n := (E-I)^{(n,q)} = \Delta^{(n,q)}$. In full, this says

$$\Delta_q^n := \begin{cases} I, & n = 0; \\ (E-I)(E-q) \cdots (E-q^{n-1}), & n \geq 1, \end{cases}$$

so

$$(2.10) \quad \Delta_q^m \binom{x}{n}_q = \begin{cases} q^{m(x+m-n)} \binom{x}{n-m}_q, & \text{if } m \leq n; \\ 0, & \text{if } m > n. \end{cases}$$

In particular, $\Delta_q^m \binom{x}{n}_q \Big|_{x=0} = \delta_{mn}$. The appearance of a function of x on the right side of (2.10), outside the q -binomial coefficient, can be removed by using an alternate q -difference operator:

$$(\mathfrak{D}_q^m f)(x) := q^{-mx} (\Delta_q^m f)(x).$$

Then

$$\mathfrak{D}_q^m \binom{x}{n}_q = \begin{cases} q^{-m(n-m)} \binom{x}{n-m}_q, & \text{if } m \leq n; \\ 0, & \text{if } m > n. \end{cases}$$

By (2.6),

$$(2.11) \quad (\Delta_q^n f)(x) = \sum_{k=0}^n \binom{n}{k}_q (-1)^k q^{k(k-1)/2} f(x+n-k).$$

The shift E commutes with multiplication by q , so Δ_q^n and $\Delta_q^{n'}$ commute, but $\Delta_q^n \Delta_q^{n'} \neq \Delta_q^{n+n'}$. To give a formula for $\Delta_q^{n+n'}$ in terms of Δ_q^n and $\Delta_q^{n'}$,

$$\Delta_q^{n+n'} = (E - q^{n+n'-1}) \cdots (E - q^{n'}) \Delta_q^{n'} = \sum_{k=0}^n \binom{n}{k}_q q^{k(k-1)/2} (-q^{n'})^k E^{n-k} \Delta_q^{n'}$$

by the q -binomial theorem, so

$$(\Delta_q^{n+n'} f)(x) = \sum_{k=0}^n \binom{n}{k}_q (-1)^k q^{k(k-1)/2} q^{n'k} (\Delta_q^{n'} f)(x+n-k) = q^{n'(n+x)} (\Delta_q^n g)(x),$$

where $g(x) = q^{-n'x} (\Delta_q^{n'} f)(x)$. This can be written more conveniently in terms of the \mathfrak{D}_q^m :

$$(2.12) \quad \mathfrak{D}_q^{n+n'} = q^{nn'} \mathfrak{D}_q^n \mathfrak{D}_q^{n'}.$$

For $n \in \mathbf{Z}$, let $\mathcal{U}_n(x) = q^{nx}$ (this depends on q), so $\Delta_q^n = \mathcal{U}_n \mathfrak{D}_q^n$ and $E^k \mathcal{U}_n = q^{kn} \mathcal{U}_n E^k$. When $q = 1$ the need for \mathcal{U}_n is not apparent. The notation \mathcal{U}_n comes from a similar function U_n used by Verdoort [25]. Her paper will be discussed in Section 4.

The effort to directly relate $\Delta_q^n \Delta_q^{n'}$ with $\Delta_q^{n+n'}$ led to a concise multiplicative relation (2.12) among the \mathfrak{D}_q 's rather than among the Δ_q 's. We now use (2.12) to give a formula for $\Delta_q^n \Delta_q^{n'}$ as a linear combination of various $\Delta_q^{n''}$, so the q -difference operators are a basis of the algebra they generate (they have no linear relations by (2.10)).

THEOREM 2.6. *For $m, n \geq 0$,*

$$\begin{aligned} \Delta_q^m \Delta_q^n &= \sum_{j=0}^m \binom{m}{j}_q (q^n - 1)(q^n - q) \cdots (q^n - q^{m-j-1}) \Delta_q^{n+j} \\ &= \sum_{j=0}^m \binom{m}{j}_q (q^n - 1)^{(m-j; q)} \Delta_q^{n+j} \\ &= \sum_{i+j=m+n} \binom{m}{i}_q \binom{n}{j}_q (q^i - 1)^{(i; q)} \Delta_q^j. \end{aligned}$$

Proof. By the q -binomial theorem,

$$\Delta_q^m \Delta_q^n = \sum_{k=0}^m \binom{m}{k}_q (-1)^{m-k} q^{(m-k)(m-k-1)/2} E^k \Delta_q^n.$$

To get a formula for $E^k \Delta_q^n$, we use the following identity: for all $k \geq 0$,

$$a^k = \sum_{i=0}^k \binom{k}{i}_q (a-1)(a-q) \cdots (a-q^{i-1}) = \sum_{i=0}^k \binom{k}{i}_q (a-1)^{(i; q)}.$$

This is dual to (2.6), or arises naturally from consideration of q -Mahler expansions in Section 3 (i.e., from the q -difference calculus), so we won't stop to motivate it here. Setting $a = E$,

$$(2.13) \quad E^k = \sum_{i=0}^k \binom{k}{i}_q \Delta_q^i.$$

Thus

$$\begin{aligned}
E^k \Delta_q^n &= E^k \mathcal{U}_n \mathcal{D}_q^n \\
&= q^{kn} \mathcal{U}_n E^k \mathcal{D}_q^n \\
&= q^{kn} \mathcal{U}_n \sum_{i=0}^k \binom{k}{i}_q \mathcal{U}_i \mathcal{D}_q^i \mathcal{D}_q^{n-i} \text{ by (2.13)} \\
&= \sum_{i=0}^k \binom{k}{i}_q q^{n(k-i)} \mathcal{U}_{n+i} \mathcal{D}_q^{n+i} \text{ by (2.12)} \\
&= \sum_{i=0}^k \binom{k}{i}_q q^{n(k-i)} \Delta_q^{n+i},
\end{aligned}$$

so

$$\begin{aligned}
\Delta_q^m \Delta_q^n &= \sum_{k=0}^m \sum_{i=0}^k (-1)^{m-k} q^{\binom{m-k}{2}} q^{n(k-i)} \binom{m}{k}_q \binom{k}{i}_q \Delta_q^{n+i} \\
&= \sum_{i=0}^m \sum_{k=0}^{m-i} (-1)^{m-i-k} q^{\binom{m-i-k}{2}} q^{nk} \binom{m-i}{k}_q \binom{m}{i}_q \Delta_q^{n+i} \\
&= \sum_{i=0}^m \binom{m}{i}_q (q^n - 1)^{(m-i; q)} \Delta_q^{n+i} \text{ by (2.6)} \\
&= \sum_{i=0}^m \binom{m}{i}_q \binom{n}{i}_q (q^i - 1)^{(i; q)} \Delta_q^{m+n-i}.
\end{aligned}$$

□

EXAMPLE. $\Delta_q^2 \Delta_q^n = (q^n - 1)(q^n - q) \Delta_q^n + (q^n - 1)(q + 1) \Delta_q^{n+1} + \Delta_q^{n+2}$.

The case $m = 1$ of Theorem 2.6 is essentially the recursive definition $\Delta_q^{n+1} = (E - q^n) \Delta_q^n$.

Once the formula in Theorem 2.6 is found, it can also be proven by induction on m , without using noncommuting operators E^k and \mathcal{U}_n , as the polynomial identity

$$\begin{aligned}
(X - 1)^{(m; q)} (X - 1)^{(n; q)} &= \sum_{i=0}^m \binom{m}{i}_q (q^n - 1)^{(m-i; q)} (X - 1)^{(n+i; q)} \\
&= \sum_{i=0}^m \binom{m}{i}_q (q^n - 1)^{(m-i; q)} (X - 1)^{(n; q)} (X - q^n)^{(i; q)}.
\end{aligned}$$

Dividing by $(X - 1)^{(n; q)}$, we get an identity which is a special case of the generalized q -binomial theorem [11, p. 252].

The q -analogue of the formula

$$\Delta^n(fg) = \sum_{k=0}^n \binom{n}{k} (\Delta^k f) (\Delta^{n-k} E^k g)$$

is

$$(2.14) \quad \Delta_q^n(fg) = \sum_{k=0}^n \binom{n}{k}_q (\Delta_q^k f) (\Delta_q^{n-k} E^k g).$$

In the inductive verification of this, use (for $r \leq n$)

$$(E - q^n)(FG) = (E - q^r)F \cdot EG + q^r F \cdot (E - q^{n-r})G$$

with $F = \Delta_q^k f$, $G = \Delta_q^{n-k} E^k g$, and $r = k$.

3. p -ADIC FEATURES OF q -FORMALISM

In Section 2, the emphasis was on q as an indeterminate. Here it will be on q as a p -adic variable, i.e., as an element of a complete valued field K containing \mathbf{Q}_p . (We do not assume $q \in \mathbf{Q}_p$.) As we will have no use for the archimedean absolute value function, the absolute value on K will be denoted simply as $|\cdot|$, and ord is the corresponding additive valuation: $|z| = (1/p)^{\text{ord}(z)}$. The valuation ring $\{z \in K : |z| \leq 1\}$ will be denoted \mathcal{O}_K , with maximal ideal \mathfrak{m}_K . We normalize the absolute value so $|p| = 1/p$.

For the benefit of readers outside of number theory, we recall some facts about power functions and roots of unity in p -adic fields.

LEMMA 3.1. (i) *The roots of unity in K which reduce to 1 in the residue field $\mathcal{O}_K/\mathfrak{m}_K$ are exactly the p th power roots of unity in K .*

(ii) *If ζ is a root of unity of order $p^N > 1$, then*

$$|\zeta - 1| = (1/p)^{1/p^{N-1}(p-1)} \geq (1/p)^{1/(p-1)}.$$

The roots of unity in K are a discrete set.

(iii) *For $q \in K$, the sequence $\{1, q, q^2, q^3, \dots\}$ can be extended to a continuous function q^x for $x \in \mathbf{Z}_p$ if and only if $|q - 1| < 1$, in which case*

$$q^x = \sum_{n \geq 0} (q - 1)^n \binom{x}{n}, \quad |q^x - 1| \leq |q - 1| < 1.$$

(iv) *If $|q - 1| < 1$, then $q^x = 1$ for $x \neq 0$ if and only if q is a root of unity of order p^N and $x \in p^N \mathbf{Z}_p$.*

Proof. (i) The residue field $\mathcal{O}_K/\mathfrak{m}_K$ has characteristic p . Since $X^a - 1$ has distinct roots in characteristic p when a is prime to p , a root of unity ζ in K of order ap^b with $a > 1$ and $(a, p) = 1$ has $\zeta^{p^b} \not\equiv 1 \pmod{\mathfrak{m}_K}$, so $\zeta \not\equiv 1 \pmod{\mathfrak{m}_K}$. Since the only p th power root of unity in characteristic p is 1, if $\zeta^{p^N} = 1$ in K , then in the residue field of K we have $\zeta^{p^N} \equiv 1 \pmod{\mathfrak{m}_K}$, so $\zeta \equiv 1 \pmod{\mathfrak{m}_K}$.

(ii) We have

$$\prod_{\substack{i=1 \\ (p,i)=1}}^{p^N} (1 - \zeta^i) = \Phi_{p^N}(1) = p,$$

so

$$p = (1 - \zeta)^{p^{N-1}(p-1)} \prod_{\substack{i=1 \\ (p,i)=1}}^{p^N} \frac{1 - \zeta^i}{1 - \zeta},$$

and for i prime to p , the ratio $(1 - \zeta^i)/(1 - \zeta) = 1 + \zeta + \cdots + \zeta^{i-1}$ is congruent in the residue field of K to $i \not\equiv 0 \pmod{\mathfrak{m}_K}$, so this ratio has absolute value 1, hence $1 - \zeta$ has the indicated size.

For two distinct roots of unity ζ and ζ' in K , either $\zeta \not\equiv \zeta' \pmod{\mathfrak{m}_K}$, so $|\zeta - \zeta'| = 1$, or $\zeta/\zeta' \equiv 1 \pmod{\mathfrak{m}_K}$, and then $|\zeta - \zeta'| = |\zeta/\zeta' - 1| \geq (1/p)^{1/(p-1)}$, so the roots of unity in K are a (bounded) discrete set.

(iii) For “if”, we have for any $m \in \mathbf{N}$ that

$$q^m = (1 + q - 1)^m = \sum_{n=0}^m (q - 1)^n \binom{m}{n}.$$

Since $(q - 1)^n \rightarrow 0$, the continuous function

$$q^x = \sum_{n \geq 0} (q - 1)^n \binom{x}{n}$$

on \mathbf{Z}_p is the p -adic interpolation of $\{q^m\}_{m \geq 0}$. For “only if”, $q^{p^N} \rightarrow q^0 = 1$ as $N \rightarrow \infty$, so $|q| = 1$ and as in (i) we conclude $|q - 1| < 1$.

(iv) Let $x = p^N u$ with u a unit in \mathbf{Z}_p . Then $q^{p^N u} = 1$ if and only if $q^{p^N} = 1$, by taking the $(1/u)$ th power. \square

Applying (iii) to q -analogues, $(m)_q = (q^m - 1)/(q - 1)$ for $m \in \mathbf{Z}$ extends to a continuous function $(x)_q$ for $x \in \mathbf{Z}_p$ if and only if $|q - 1| < 1$, in which case the extension to \mathbf{Z}_p is

$$(x)_q = \begin{cases} \frac{q^x - 1}{q - 1}, & \text{if } q \neq 1; \\ x, & \text{if } q = 1, \end{cases}$$

and by (iii), $(x)_q \equiv x \pmod{\mathfrak{m}_K}$. In particular, if $x \in \mathbf{Z}_p^\times$, then $(x)_q \in \mathcal{O}_K^\times$.

For $q \neq 1$, $(x)_q$ is a nonvanishing function unless, by (iv), q is a nontrivial root of unity of order p^N , in which case $(x)_q = (j)_q$ where $x \equiv j \pmod{p^N}$ and $0 \leq j \leq p^N - 1$.

We now define the q -analogue of binomial coefficient functions.

For $|q - 1| < 1$, $\binom{m}{n}_q$ has a continuous extension from $m \in \mathbf{Z}$ to $x \in \mathbf{Z}_p$, given by

$$\begin{aligned} \binom{x}{n}_q &= \frac{(x)_q (x - 1)_q \cdots (x - n + 1)_q}{(n)_q!} \\ &= \frac{(q^x - 1)(q^{x-1} - 1) \cdots (q^{x-n+1} - 1)}{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}, \end{aligned}$$

provided $(n)_q! \neq 0$, i.e., q is not a nontrivial p th power root of unity of order $\leq n$.

If $|q - 1| < 1$ and q is a root of unity of order p^N , Corollary 2.1 implies $\binom{x}{n}_q$ is a polynomial function of x on cosets of $p^N \mathbf{Z}_p$. For $x = p^N y + r$ and $n = p^N l + s$ where $0 \leq r, s < p^N$, Theorem 2.5(ii) extends by continuity to

$$(3.1) \quad \binom{x}{n}_q = \binom{y}{l} \binom{r}{s}_q.$$

For example, if $p = 2$, then

$$\binom{x}{2l}_{-1} = \begin{cases} \binom{x/2}{l}, & \text{if } x \equiv 0 \pmod{2}; \\ \binom{(x-1)/2}{l}, & \text{if } x \equiv 1 \pmod{2}, \end{cases} \quad \binom{x}{2l+1}_{-1} = \begin{cases} 0, & \text{if } x \equiv 0 \pmod{2}; \\ \binom{(x-1)/2}{l}, & \text{if } x \equiv 1 \pmod{2}. \end{cases}$$

So $\binom{x}{n}_q$ is an exponential function of x (a polynomial in q^x) if q is not a root of unity and is locally a polynomial in x if q is a root of unity.

By Theorem 2.1, $|\binom{x}{n}_q| \leq 1$ for all $x \in \mathbf{Z}_p$, with equality if $x = n$.

The difference operators Δ_q^n and \mathfrak{D}_q^n make sense on functions of a p -adic integer variable x , and equations (2.10) and (2.11) remain true when x is any p -adic integer.

By continuity, Theorems 2.3 and 2.4 become

THEOREM 3.1. *If $|q - 1| < 1$, then for all $x, y \in \mathbf{Z}_p$, $\binom{x+y}{k}_q = \sum_{j=0}^k \binom{x}{j}_q \binom{y}{k-j}_q q^{j(y-(k-j))}$.*

THEOREM 3.2. *If $x \in \mathbf{Z}_p$ and $|q_1 - 1| < 1$, $|q_2 - 1| < 1$, then $|\binom{x}{n}_{q_1} - \binom{x}{n}_{q_2}| \leq |q_1 - q_2|$.*

So $\binom{x}{n}_q = \lim_{q' \rightarrow q} \binom{x}{n}_{q'}$. In particular, formulas involving q -binomial coefficients when q is a root of unity can be computed first at non roots of unity and then pass to a limit.

For example, let $1 \leq k \leq p^r$ with $k = p^j k'$ and k' prime to p . For $|q - 1| < 1$ with q not a root of unity,

$$\binom{p^r}{k}_q = \frac{(p^r)_q}{(k)_q} \binom{p^r - 1}{k - 1}_q = (p^{r-j})_{q^{p^j}} \frac{1}{(k')_{q^{p^j}}} \binom{p^r - 1}{k - 1}_q.$$

In $\mathcal{O}_K/\mathfrak{m}_K$, $\binom{p^r-1}{k-1}_q \equiv \binom{p^r-1}{k-1} \equiv (-1)^{k-1}$ and $(k')_{q^{p^j}} \equiv k' \not\equiv 0$, so

$$(3.2) \quad \left| \binom{p^r}{k}_q \right| = |(p^{r-j})_{q^{p^j}}|.$$

By continuity in q , (3.2) is also true when q is a root of unity. Alternatively, (3.1) could be used instead for a direct calculation when q is not a root of unity.

We now discuss the q -analogue of Mahler expansions.

THEOREM 3.3 (q -MAHLER THEOREM). *For $q \in K$ with $|q - 1| < 1$, every continuous function $f: \mathbf{Z}_p \rightarrow K$ has a unique representation in the form*

$$f(x) = \sum_{n \geq 0} c_{n,q} \binom{x}{n}_q$$

where $c_{n,q} \in K$ and $\lim_{n \rightarrow \infty} c_{n,q} = 0$. A formula for $c_{n,q}$ is

$$\begin{aligned} c_{n,q} &= (\Delta_q^n f)(0) \\ &= \sum_{k=0}^n \binom{n}{k}_q (-1)^k q^{k(k-1)/2} f(n-k) \\ &= \sum_{k=0}^n \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k). \end{aligned}$$

We will give four proofs of Theorem 3.3 below.

In Theorem 3.3, we call $c_{n,q}$ the n th q -Mahler coefficient of f and $\sum c_{n,q} \binom{x}{n}_q$ the q -Mahler expansion of f . The terms ‘‘Mahler coefficient’’ and ‘‘Mahler expansion’’ will refer to the case $q = 1$. The formula for $c_{n,q}$ in Theorem 3.3 will be called the q -Mahler Inversion Formula.

The formula for $c_{n,q}$ follows from computing $(\Delta_q^n f)(0)$ using (2.10). Replacing f by $(E^y f)(x) = f(x+y)$, we have $\lim_{n \rightarrow \infty} (\Delta_q^n f)(y) = 0$ for all $y \in \mathbf{Z}_p$. Like the case $q = 1$, this limit turns out to be uniform in y , and in fact there is some uniformity in q as well (which

is not apparent by looking only at the case $q = 1$). Such uniformities will arise from two of the proofs of Theorem 3.3.

EXAMPLE. For $|a - 1| < 1$ and $|q - 1| < 1$,

$$(3.3) \quad a^x = \sum_{n \geq 0} (a - 1)(a - q) \cdots (a - q^{n-1}) \binom{x}{n}_q = \sum_{n \geq 0} (a - 1)^{(n; q)} \binom{x}{n}_q.$$

EXAMPLE. Using the q -binomial theorem, the sequence $(1 + t)^{(m; q)}$ extends continuously from $m \in \mathbf{N}$ to $x \in \mathbf{Z}_p$ if and only if $|t| < 1$, when

$$(1 + t)^{(x; q)} = \sum_{n \geq 0} q^{n(n-1)/2} t^n \binom{x}{n}_q.$$

This could also be proven in a style similar to that of Lemma 3.1 (iii).

For any $x, y \in \mathbf{Z}_p$, $(1 + t)^{(x+y; q)} = (1 + t)^{(x; q)}(1 + q^x t)^{(y; q)}$. Setting $y = -x$ yields

$$((1 + t)^{(x; q)})^{-1} = (1 + q^x t)^{(-x; q)}.$$

For example, computing $(1 + q^m t)^{(-m; q)}$ in two ways for $m \geq 1$, we have

$$\frac{1}{(1 + t)(1 + qt) \cdots (1 + q^{m-1}t)} = \sum_{n \geq 0} q^{n(n-1)/2} (q^m t)^n \binom{-m}{n}_q = \sum_{n \geq 0} \binom{m + n - 1}{n}_q (-t)^n,$$

which is due to Cauchy [4, Eq. 19, p. 46] as an identity over the complex numbers.

WARNING. For $|a - 1| < 1$, writing $a = 1 + t$, it seems reasonable to define $a^{(x; q)} = (1 + t)^{(x; q)}$ in the sense of the above example. However, although $|a^{(m; q)} - 1| < 1$ and $(1 + T)^{(mn; q)} = ((1 + T)^{(m; q)})^{(n; q^m)}$ (which implies (2.1) by looking at the coefficient of T), it is false that $a^{(mn; q)} = (a^{(m; q)})^{(n; q^m)}$, even when $m = n = 2$. A correct way to state the q -version of $(1 + T)^{mn} = ((1 + T)^m)^n$ so that it is valid to specialize the variable is

$$(1 + T)^{(mn; q)} = (1 + T)^{(n; q^m)}(1 + qT)^{(n; q^m)} \cdots (1 + q^{m-1}T)^{(n; q^m)}.$$

Our first proof of Theorem 3.3 will deduce the result from the known case $q = 1$. Recall that a countable set of vectors $\{e_n\}_{n \geq 0}$ in a K -Banach space $(V, \|\cdot\|)$ (we assume the norm on V is nonarchimedean: $\|v + w\| \leq \max(\|v\|, \|w\|)$) is called an *orthonormal basis* if every $v \in V$ has a unique representation in the form $v = \sum c_n e_n$ where $c_n \rightarrow 0$ and $\|v\| = \max |c_n|$. Mahler's theorem says the functions $\binom{x}{n}$ are an orthonormal basis of $C(\mathbf{Z}_p, K)$, topologized by the sup-norm.

The following standard lemma shows that a small perturbation of an orthonormal basis is still an orthonormal basis. The ideas in the proof are taken from [3, Prop. 2 §1.1.4, Prop. 4 §2.7.2].

LEMMA 3.2. *Let K be a complete nonarchimedean nontrivially valued field and V be a K -Banach space with an orthonormal basis $\{e_n\}_{n \geq 0}$. If $e'_n \in V$ with $\sup_{n \geq 0} \|e_n - e'_n\| < 1$, then $\{e'_n\}$ is an orthonormal basis of V .*

Proof. Step 1) $\|\sum_{n=0}^N c_n e'_n\| = \max_{0 \leq n \leq N} |c_n|$.

Let $\varepsilon = \sup_{n \geq 0} \|e_n - e'_n\| < 1$. Writing

$$\sum_{n=0}^N c_n e'_n = \sum_{n=0}^N c_n (e'_n - e_n) + \sum_{n=0}^N c_n e_n,$$

the first sum has size at most $\varepsilon \max |c_n|$.

Step 2) The K -linear span (= finite linear combinations) of the e'_n is dense in V .

Let W be this span. For $v \in V$, let $v = \sum_{n \geq 0} c_n e_n$. Choose N so $|c_n| \leq \varepsilon \|v\|$ for $n \geq N+1$. Then

$$v - \sum_{n=0}^N c_n e'_n = \sum_{n=0}^N c_n (e_n - e'_n) + \sum_{n \geq N+1} c_n e_n$$

has norm $\leq \varepsilon \|v\|$. Assume W is not dense, so there is $v \in V$ such that $a = \inf_{w \in W} \|v - w\| > 0$. Since $a/\varepsilon > a$, there is $w \in W$ such that $0 < \|v - w\| < a/\varepsilon$. From above, there is $w' \in W$ such that

$$\|v - w - w'\| \leq \varepsilon \|v - w\| < a,$$

a contradiction.

Step 3) $\{e'_n\}$ is an orthonormal basis.

By Step 1, it suffices to show for each $v \in V$ that $v = \sum c_n e'_n$ for some sequence $c_n \rightarrow 0$ in K .

Choose $w_1 \in W$ such that $\|v - w_1\| \leq 1/2$. Choose $w_2 \in W$ such that $\|v - w_1 - w_2\| \leq 1/4$. Continuing, choose $w_m \in W$ such that $\|v - w_1 - \dots - w_m\| \leq 1/2^m$. Then $\|w_m\| \rightarrow 0$ and $v = \sum w_m$. Writing $w_m = \sum_n b_{m,n} e'_n$, we have $b_{m,n} = 0$ for n large and $|b_{m,n}| \leq \|w_m\|$ by Step 1. Thus

$$v = \sum_m \left(\sum_n b_{m,n} e'_n \right) = \sum_n \left(\sum_m b_{m,n} \right) e'_n,$$

where the interchange of the double sum is justified by [12, Lemma 4.1.3]. \square

Here is a first proof of Theorem 3.3.

Proof. By Mahler's theorem, $\left\{ \binom{x}{n} \right\}_{n \geq 0}$ is an orthonormal basis of $C(\mathbf{Z}_p, K)$. For all $n \geq 0$, Theorem 3.2 implies

$$\left| \binom{x}{n}_q - \binom{x}{n} \right|_{\sup} \leq |q - 1| < 1.$$

Therefore we are done by Lemma 3.2. \square

This proof of Theorem 3.3 is succinct, but depends on already having the result in the case $q = 1$. The same argument would deduce the result for all q with $|q - 1| < 1$ if we had it for any one such q .

By a similar idea, since $\left\{ \binom{x}{m} \binom{y}{n} \right\}$ is an orthonormal basis of $C(\mathbf{Z}_p \times \mathbf{Z}_p, K)$, topologized by the sup-norm, so is $\left\{ \binom{x}{m}_{q_1} \binom{y}{n}_{q_2} \right\}$ for fixed $q_1, q_2 \in K$ with $|q_1 - 1|, |q_2 - 1| < 1$. There is a similar extension to $C(\mathbf{Z}_p^r, K)$ for any $r \geq 1$.

Since $(\Delta_q^n f)(x) = \Delta_q^n (E^x f)(0)$, by the q -Mahler theorem we have $\lim_{n \rightarrow \infty} (\Delta_q^n f)(x) = 0$ for each $x \in \mathbf{Z}_p$. However, this limit is actually uniform in x . To see this we give a second proof of the q -Mahler theorem, one which will not assume Mahler's theorem already. It will show directly that $\lim_{n \rightarrow \infty} \Delta_q^n f = 0$ in $C(\mathbf{Z}_p, K)$.

First we record a lemma. It gives some properties of the size of $(x)_q$. Extending (2.1) from \mathbf{Z} to \mathbf{Z}_p , if $|q - 1| < 1$ then $(xy)_q = (x)_q(y)_{q^x}$ for $x, y \in \mathbf{Z}_p$. In particular, for $n \in \mathbf{N}$ and $u \in \mathbf{Z}_p^\times$,

$$(3.4) \quad (p^n u)_q = (p^n)_q (u)_{q^{p^n}}.$$

LEMMA 3.3. *Let $|q - 1| < 1$.*

- (i) *If $x = p^n u$ with $u \in \mathbf{Z}_p^\times$, $|(x)_q| = |(p^n)_q|$.*
- (ii) *$|(p^n)_q| \leq \prod_{i=0}^{n-1} \max(|q^{p^i} - 1|, 1/p) \leq \max(|q - 1|, 1/p)^n < 1$.*
- (iii) *If $|q - 1| < (1/p)^{1/(p-1)}$, then $|(x)_q| = |x|$ for all $x \in \mathbf{Z}_p$.*

Proof. (i) Use (3.4), recalling $(u)_{q^{p^n}} \equiv u \not\equiv 0 \pmod{\mathfrak{m}_K}$.

(ii) By (2.1),

$$(3.5) \quad (p^n)_q = (p)_q (p)_{q^p} \cdots (p)_{q^{p^{n-1}}},$$

so it suffices to show for $|q - 1| < 1$ that $|(p)_q| \leq \max(|q - 1|, 1/p)$. In $\mathcal{O}_K/(q - 1, p)$,

$$(p)_q = \Phi_p(q) \equiv (q - 1)^{p-1} \equiv 0.$$

(iii) By (i), we only need to show the result for $x = p^n$. Moreover, by (3.5) and $|q^{p^i} - 1| \leq |q - 1| < (1/p)^{1/(p-1)}$, it suffices to show the result for $x = p$. Since

$$(p)_q = \frac{q^p - 1}{q - 1} = \sum_{k=1}^p \binom{p}{k} (q - 1)^{k-1}$$

and each term in the sum except the one for $k = 1$ has size less than $1/p$, we're done. \square

As a consequence of (i) and (ii), we have

$$|(x)_q - (y)_q| = |(x - y)_q| \leq \max(|q - 1|, 1/p)^{\text{ord}(x-y)},$$

which can be rewritten as $|q^x - q^y| \leq |q - 1| \max(|q - 1|, 1/p)^{\text{ord}(x-y)}$, in which form it appears in [22, Theorem 32.4].

From (ii), (3.2) can be weakened to

$$(3.6) \quad \left| \binom{p^r}{k}_q \right| \leq \max(|q - 1|, 1/p)^{r-j},$$

where we recall $1 \leq k \leq p^r$, $j = \text{ord}(k)$.

We now give a second proof of Theorem 3.3. The idea is taken from the proof of Mahler's theorem in [22, Exer. 52.E].

Proof. Since $|\Delta_q^{n+1} f|_{\text{sup}} \leq |\Delta_q^n f|_{\text{sup}}$, it suffices to show $\lim_{r \rightarrow \infty} \Delta_q^{p^r} f = 0$. We have

$$\begin{aligned} (\Delta_q^{p^r} f)(x) &= \sum_{k=0}^{p^r} \binom{p^r}{k}_q (-1)^{p^r-k} q^{(p^r-k)(p^r-k-1)/2} f(k+x) \\ &= \sum_{k=0}^{p^r} \binom{p^r}{k}_q (-1)^{p^r-k} q^{(p^r-k)(p^r-k-1)/2} (f(k+x) - f(x)). \end{aligned}$$

The $k = 0$ term vanishes, so by (3.6)

$$|\Delta_q^{p^r} f|_{\text{sup}} \leq \max_{i+j=r} \max(|q - 1|, 1/p)^i \rho_j(f),$$

where $\rho_j(f) = \sup_{|x-y| \leq 1/p^j} |f(x) - f(y)|$. The terms indexed by i and j are both uniformly bounded above, and each tends to zero for large values of the index. \square

Not only does this show $\lim_{n \rightarrow \infty} (\Delta_q^n f)(x) = 0$ uniformly in x , but also (for fixed $\delta \in (0, 1)$) uniformly in q for $|q - 1| \leq \delta < 1$.

For the third proof of the q -Mahler theorem, we extend a periodicity property of ordinary binomial coefficients to q -binomial coefficients: for any $N \geq 1$ and all $n < p^N$,

$$a \equiv b \pmod{p^N} \Rightarrow \binom{a}{n} \equiv \binom{b}{n} \pmod{p}.$$

For q -binomial coefficients, the same result is true provided N is taken large enough depending on q .

LEMMA 3.4. *Let $|q - 1| < 1$. For N large, depending on q , if $x \equiv y \pmod{p^N \mathbf{Z}_p}$ and $n < p^N$ then*

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \frac{1}{p}.$$

More precisely, this is true if $1/(p^{N-1}(p-1)) < \text{ord}(q-1)$.

Proof. By Theorem 2.5,

$$m_1 \equiv m_2 \pmod{p^N} \Rightarrow \binom{m_1}{n}_q - \binom{m_2}{n}_q \in \Phi_{p^N}(q) \mathbf{Z}[q].$$

So by continuity,

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq |\Phi_{p^N}(q)| = |(p)_{q^{p^{N-1}}}|.$$

For N large, $|q^{p^{N-1}} - 1| < (1/p)^{1/(p-1)}$, so $(p)_{q^{p^{N-1}}}$ has size $|p| = 1/p$ by Lemma 3.3(iii).

Let's be more precise about how large N has to be. For any N ,

$$\Phi_{p^N}(q) = \prod_{\substack{\zeta^{p^N}=1 \\ \zeta^{p^{N-1}} \neq 1}} (q - \zeta).$$

There are $p^{N-1}(p-1)$ terms in the product. When $1/p^{N-1}(p-1) < \text{ord}(q-1)$, then $|q-1| < |\zeta-1| = (1/p)^{1/p^{N-1}(p-1)}$ for all such ζ by Lemma 3.1(ii), so all the terms have the same size and therefore

$$|\Phi_{p^N}(q)| = \frac{1}{p}.$$

\square

If we work modulo $(q-1, p)$, then for $x \equiv y \pmod{p^N}$ and $n < p^N$, $\binom{x}{n}_q \equiv \binom{x}{n} \equiv \binom{y}{n} \equiv \binom{y}{n}_q$, so without needing N to be large, we have $|\binom{x}{n}_q - \binom{y}{n}_q| \leq \max(|q-1|, 1/p)$.

Now we give a third proof of Theorem 3.3. Like the second, it does not require prior knowledge at $q = 1$. It is based on the proof in [17, pp. 99-100].

Proof. Let

$$L: \{(c_n)_{n \geq 0} : c_n \in K, c_n \rightarrow 0\} \rightarrow C(\mathbf{Z}_p, K)$$

by $(c_n) \mapsto \sum_{n \geq 0} c_n \binom{x}{n}_q$. This is K -linear and continuous, where the domain and range are both topologized by the appropriate sup-norm. We want to show L is onto. By scaling it suffices to show the restriction $L: B \rightarrow C(\mathbf{Z}_p, \mathcal{O}_K)$ is onto, where

$$B = \{(c_n) : |c_n| \leq 1, c_n \rightarrow 0\}.$$

By completeness of B and continuity of L , it is enough to show that for any $f \in C(\mathbf{Z}_p, \mathcal{O}_K)$, there is some $s \in B$ such that $|f - L(s)| \leq |p|$. (Then apply the result to $g = (f - L(s))/p$ to get $s' \in B$ such that $|f - L(s + ps')| \leq |p^2|$, etc.) That is, we want to show surjectivity of the map

$$\{(c_n) : c_n \in \mathcal{O}_K/p, c_n = 0 \text{ for large } n\} \rightarrow C(\mathbf{Z}_p, \mathcal{O}_K/p)$$

given by

$$(3.7) \quad (c_n) \mapsto \sum_{n \geq 0} c_n \binom{x}{n}_q \pmod{p}.$$

Note that the quotient topology on \mathcal{O}_K/p is the discrete topology. Thus

$$(3.8) \quad C(\mathbf{Z}_p, \mathcal{O}_K/p) = \bigcup_{N \geq 1} \text{Maps}(\mathbf{Z}_p/p^N \mathbf{Z}_p, \mathcal{O}_K/p).$$

The union in (3.8) can be taken over just large integers. Lemma 3.4 suggests that at least for large N (depending on q), $f \in C(\mathbf{Z}_p, \mathcal{O}_K/p)$ factors through $\mathbf{Z}_p/p^N \mathbf{Z}_p$ when its n th q -Mahler coefficient vanishes for $n \geq p^N$, thus suggesting the more precise surjectivity of

$$(3.9) \quad \{(c_n)_{n=0}^{p^N-1} : c_n \in \mathcal{O}_K/p\} \rightarrow \text{Maps}(\mathbf{Z}_p/p^N \mathbf{Z}_p, \mathcal{O}_K/p)$$

given by (3.7) with the sum over $0 \leq n \leq p^N - 1$. (Note that by Lemma 3.4, $\binom{x}{n}_q \pmod{p}$ is well-defined on $\mathbf{Z}_p/p^N \mathbf{Z}_p$ for N large and $n < p^N$.) The surjectivity (even bijectivity) of (3.9) follows from the argument that q -Mahler coefficients are unique. \square

We could have worked in $\mathcal{O}_K/(q-1, p)$ and not needed to use only large N at the end of the proof.

Here's a fourth proof of Theorem 3.3, which like the second will yield some uniformity statements in q .

Proof. Define the numbers $c_n = c_{n,q}$ as in the statement of Theorem 3.3, so

$$f(m) = \sum_{n \geq 0} c_n \binom{m}{n}_q$$

for all nonnegative integers m . We thus only need to show that $|c_n| \rightarrow 0$. To do this we adapt Bojanic's argument in [2].

Bojanic's proof uses two different formulas for $(\Delta^n f)(m)$. First,

$$(\Delta^n f)(m) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(k+m).$$

Writing $(\Delta^n f)(m) = (\Delta^n E^m f)(0)$, we also have

$$E^m = (I + \Delta)^m = \sum_{j=0}^m \binom{m}{j} \Delta^j \Rightarrow (\Delta^n f)(m) = \sum_{j=0}^m \binom{m}{j} (\Delta^{n+j} f)(0).$$

For the q -analogue of these, (2.11) gives

$$(\Delta_q^n f)(m) = \sum_{k=0}^n \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k+m),$$

while (2.13) gives

$$E^m = \sum_{j=0}^m \binom{m}{j}_q \Delta_q^j \Rightarrow (\Delta_q^n f)(m) = (\Delta_q^n E^m f)(0) = \sum_{j=0}^m \binom{m}{j}_q q^{n(m-j)} (\Delta_q^{n+j} f)(0).$$

Equating these formulas for $(\Delta_q^n f)(m)$ and isolating the $j = m$ term,

$$c_{n+m} = \sum_{k=0}^n \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k+m) - \sum_{j=0}^{m-1} \binom{m}{j}_q q^{(m-j)n} c_{n+j}.$$

With this formula we show $|c_n| \rightarrow 0$.

The $j = 0$ term is $q^{mn} c_n = q^{mn} \sum_{k=0}^n \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k)$, so

$$c_{n+m} = \sum_{k=0}^n \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} (f(k+m) - q^{mn} f(k)) - \sum_{j=1}^{m-1} \binom{m}{j}_q q^{(m-j)n} c_{n+j}.$$

Scaling, we may assume $|f(x)| \leq 1$ for all $x \in \mathbf{Z}_p$, so $|c_n| \leq 1$ for all n .

Let $m = p^r$, for r to be determined. Then

$$|c_{n+p^r}| \leq \max_{\substack{0 \leq k \leq n \\ 1 \leq j \leq p^r-1}} \left\{ |f(k+p^r) - q^{p^r n} f(k)|, \left| \binom{p^r}{j}_q c_{n+j} \right| \right\}.$$

For such j , $\left| \binom{p^r}{j}_q \right| \leq |\Phi_{p^r}(q)|$ by (2.7).

Choose $\varepsilon > 0$. For large r , depending on f ,

$$|x - y| \leq \frac{1}{p^r} \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

Thus $f(k+p^r) - q^{p^r n} f(k) = f(k+p^r) - f(k) + f(k)(1 - q^{p^r n})$, where the first term has size at most ε , while the second is at most $|q-1| \max(|q-1|, 1/p)^r$, which is $\leq \varepsilon$ for r large (depending on q).

By Lemma 3.4, $|\Phi_{p^r}(q)| = 1/p$ for all large r , depending on q . So there is a large r such that for all $n \geq 0$,

$$\begin{aligned} |c_{n+p^r}| &\leq \max_{1 \leq j \leq p^r-1} (\varepsilon, (1/p)|c_{n+j}|) \\ &\leq \max(\varepsilon, 1/p). \end{aligned}$$

Thus $|c_n| \leq \max(\varepsilon, 1/p)$ for $n \geq p^r$. Replacing n by $n + p^r$ gives, for all $n \geq 0$,

$$\begin{aligned} |c_{n+2p^r}| &\leq \max_{1 \leq j \leq p^r-1} (\varepsilon, (1/p)|c_{n+p^r+j}|) \\ &\leq \max(\varepsilon, 1/p^2). \end{aligned}$$

So

$$|c_n| \leq \max(\varepsilon, 1/p^2)$$

for $n \geq 2p^r$. Repeating this $s-1$ more times gives

$$|c_n| \leq \max(\varepsilon, 1/p^s)$$

for $n \geq sp^r$. Choosing s so large that $1/p^s \leq \varepsilon$ we have $|c_n| \leq \varepsilon$ if $n \geq sp^r$. \square

Since the functions $E^x f$ are equicontinuous, this proof shows $\lim_{n \rightarrow \infty} \Delta_q^n f = 0$ uniformly in q for $|q - 1| \leq \delta < 1$.

For the reader who knows about q -derivatives, a second way to obtain the two formulas for $(\Delta_q^n f)(m)$ in the proof above is to make the proposed equality of these two expressions a universal polynomial identity, and establish it by q -differentiating the equation

$$\sum_{k \geq 0} f(k) \frac{X^k}{(k)_q!} = E_q(X) \sum_{n \geq 0} c_n \frac{X^n}{(n)_q!}$$

m times, dividing by $E_q(X)$, and then equating coefficients of X^n .

Although the q -Mahler expansion is treated above for a single function $f \in C(\mathbf{Z}_p, K)$, we will look in Section 5 at an example of a family of functions $f_q \in C(\mathbf{Z}_p, K)$ that depends continuously on q and consider the expansion of f_q relative to the q -Mahler basis.

4. PROPERTIES OF q -MAHLER EXPANSIONS

We now go through properties of q -Mahler expansions that are analogous to properties of Mahler expansions. *Throughout this section, $|q - 1| < 1$.*

First, note that $\{q \in K : |q - 1| < 1\}$ is a multiplicative group, unlike the parameter set that arises for q -series over \mathbf{C} , the open unit disk. So we can also consider $1/q$ -Mahler expansions.

THEOREM 4.1. *Let $|q - 1| < 1$, $f \in C(\mathbf{Z}_p, K)$ with q -Mahler coefficients $c_{n,q}$. Then*

- (i) $\sup_{x \in \mathbf{Z}_p} |f(x)| = \max_{n \geq 0} |c_{n,q}|$.
- (ii) $f(x + 1) = \sum_{n \geq 0} (q^n c_{n,q} + c_{n+1,q}) \binom{x}{n}_q$.
- (iii) $f(x + y) = \sum_{n \geq 0} (\Delta_q^n f)(y) \binom{x}{n}_q$.
- (iv) $f(-x) = \sum_{n \geq 0} c_{n,q} (-1)^n q^{-n(n+1)/2} \binom{x+n-1}{n}_{1/q}$.
- (v) $(x)_q f(x) = \sum_{n \geq 1} (n)_q (c_{n,q} + q^{n-1} c_{n-1,q}) \binom{x}{n}_q$.

Proof. Part (i) follows from the q -Mahler Inversion Formula, or from the first proof of Theorem 3.3.

Part (ii) is a special case of part (iii) or can be done on its own. For part (iii), note $f(x + y) = (E^y f)(x)$ and the n th q -Mahler coefficient of $E^y f$ is $(\Delta_q^n (E^y f))(0) = (\Delta_q^n f)(y)$.

Part (iii) can also be proven by using the q -Vandermonde formula and an interchange of a double sum, which is Mahler's original method at $q = 1$.

For part (iv), use (2.5). Note that the expansion given in (iv) is related to a $1/q$ -Mahler expansion, which can be explicitly computed using Theorem 3.1.

For part (v), use $(n)_q \binom{x}{n}_q = (x)_q \binom{x-1}{n-1}_q$.

\square

In light of (iii), $(\Delta_q^n f)(y)$ should be called the n th q -Mahler coefficient of f at y .

As with Mahler expansions, a function $\mathbf{Z}_p \rightarrow K$ with a pointwise representation as $\sum c_{n,q} \binom{x}{n}_q$ must be continuous, since $c_{n,q} \rightarrow 0$ by looking at $x = -1$.

Let's see how the difference operators act on q -Mahler expansions. For $q = 1$,

$$\Delta^m \left(\sum_{j \geq 0} c_j \binom{x}{j} \right) = \sum_{j \geq 0} c_{m+j} \binom{x}{j},$$

but for general q (2.10) implies

$$(4.1) \quad \Delta_q^m \left(\sum_{j \geq 0} c_{j,q} \binom{x}{j}_q \right) = \sum_{j \geq 0} c_{m+j,q} q^{m(x-j)} \binom{x}{j}_q,$$

which is not a q -Mahler expansion, because of the term q^{mx} . So using the operator $(\mathfrak{D}_q^m f)(x) = q^{-mx} (\Delta_q^m f)(x)$, we can write this instead as

$$\mathfrak{D}_q^m \left(\sum_{j \geq 0} c_{j,q} \binom{x}{j}_q \right) = \sum_{j \geq 0} c_{m+j,q} q^{-mj} \binom{x}{j}_q.$$

The formula in part (v) of Theorem 4.1 can be extended to $\binom{x}{m}_q f(x)$, computing the n th q -Mahler coefficient by (2.14) and (4.1) for $n \geq m$:

$$\begin{aligned} \Delta_q^n \left(\binom{x}{m}_q f(x) \right) (0) &= \sum_{k=0}^n \binom{n}{k}_q \left(\Delta_q^k \binom{x}{m}_q \right) (0) \left(\Delta_q^{n-k} f \right) (k) \\ &= \binom{n}{m}_q \left(\Delta_q^{n-m} f \right) (m) \\ &= \binom{n}{m}_q \sum_{k=0}^m q^{(n-m)k} \binom{m}{k}_q c_{n-k,q}. \end{aligned}$$

We now discuss the relation between differentiability and q -Mahler expansions. When $q = 1$, Mahler shows in [18, Theorem 3], [19] that $f \in C(\mathbf{Z}_p, K)$ is differentiable at y if and only if $\lim_{m \rightarrow \infty} (\Delta^m f)(y)/m = 0$ and then

$$(4.2) \quad f'(y) = \sum_{m \geq 1} \frac{(\Delta^m f)(y)}{m} (-1)^{m-1}.$$

The extension of this result to general $|q - 1| < 1$ involves the p -adic logarithm, whose properties we will summarize for the convenience of readers outside of number theory. These readers should notice in particular part (iv) below, which says the p -adic logarithm is locally an isometry.

LEMMA 4.1. (i) *The series $\log_p(1+z) = \sum_{n \geq 1} (-1)^{n-1} z^n/n$ converges at $z \in K$ if and only if $|z| < 1$.*

(ii) *If $|u_1 - 1|, |u_2 - 1| < 1$, then $\log_p(u_1 u_2) = \log_p(u_1) + \log_p(u_2)$.*

(iii) *For $|q - 1| < 1$, $\lim_{x \rightarrow 0} \frac{q^x - 1}{x} = \log_p q$.*

(iv) *If $|u - v| < (1/p)^{1/(p-1)}$, then $|\log_p u - \log_p v| = |u - v|$.*

(v) *$\log_p u = 0$ if and only if u is a p th power root of unity in K .*

(vi) *If $|\zeta - 1| < 1$ and $\zeta^m = 1$, then $\lim_{q \rightarrow \zeta} \frac{\log_p q}{q^m - 1} = \frac{1}{m}$.*

Proof. i) $|z|^n \leq |z^n/n| \leq n|z|^n$.

ii) See [12, Prop. 4.5.3].

iii) For $x \neq 0$, $(q^x - 1)/x = \sum_{n \geq 1} ((q - 1)^n/n) \binom{x-1}{n-1}$ and $(q - 1)^n/n \rightarrow 0$ by (i).

iv) By (ii) we may take $v = 1$. The first term of the series for $\log_p u$ is $u - 1$. For $u \neq 1$, all the remaining terms have size less than $|u - 1|$ since for $n \geq 2$, the unique minimum of $|n|^{1/(n-1)} = (1/p)^{\text{ord}(n)/(n-1)}$ occurs at $n = p$.

v) For any integer r , $\log_p u = 0$ if and only if $\log_p(u^{p^r}) = 0$. For r large, $|u^{p^r} - 1| < (1/p)^{1/(p-1)}$, and by (iv) the only z with $|z - 1| < (1/p)^{1/(p-1)}$ and $\log_p z = 0$ is $z = 1$.

vi) $(\log_p q)/(q^m - 1) = \log_p(q/\zeta)/((q/\zeta)^m - 1)$ and $\lim_{u \rightarrow 1}(\log_p u)/(u^m - 1) = 1/m$ since $\lim_{u \rightarrow 1}(\log_p u)/(u - 1) = 1$ from the definition of $\log_p u$. □

LEMMA 4.2. *Let $g: \mathbf{Z}_p \rightarrow K$ be continuous on $\mathbf{Z}_p - \{-1\}$, with $g(x) = \sum_{n \geq 0} c_n \binom{x}{n}_q$ for $x \neq -1$. Then g is continuous at -1 if and only if $c_n \rightarrow 0$, in which case $g(-1) = \sum_{n \geq 0} c_n \binom{-1}{n}_q$.*

Proof. The “if” direction is clear. For “only if”, continuity of g at -1 is the same as continuity of g on \mathbf{Z}_p , by our hypothesis. Letting x run through the nonnegative integers, we see by the q -Mahler Inversion Formula that c_n is the n th q -Mahler coefficient of g , so we’re done by Theorem 3.3. □

Here is the test for differentiability with q -Mahler expansions. Compare with formulas for the derivative in (4.2).

THEOREM 4.2. *Let $f \in C(\mathbf{Z}_p, K)$.*

(i) *When q is not a (nontrivial) root of unity, f is differentiable at $x \in \mathbf{Z}_p$ if and only if $\lim_{m \rightarrow \infty}(\Delta_q^m f)(x)/(m)_q = 0$, in which case*

$$f'(x) = \frac{\log_p q}{q-1} \sum_{m \geq 1} \frac{(\Delta_q^m f)(x)}{(m)_q} (-1)^{m-1} q^{-m(m-1)/2}.$$

(ii) *When q is a root of unity of order p^N ($N \geq 0$), f is differentiable at $x \in \mathbf{Z}_p$ if and only if $\lim_{l \rightarrow \infty}(\Delta_q^{p^N l} f)(x)/p^{Nl} = 0$, in which case*

$$f'(x) = \sum_{l \geq 1} \frac{(\Delta_q^{p^N l} f)(x)}{p^{Nl}} (-1)^{l-1}.$$

Proof. (i) For $h \neq 0$, $f(x+h) = f(x) + \sum_{m \geq 1} (\Delta_q^m f)(x) \binom{h}{m}_q$ by Theorem 4.1. Therefore

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \sum_{m \geq 1} (\Delta_q^m f)(x) \frac{1}{h} \binom{h}{m}_q \\ &= \frac{(h)_q}{h} \sum_{m \geq 1} \frac{(\Delta_q^m f)(x)}{(m)_q} \binom{h-1}{m-1}_q. \end{aligned}$$

Since $(h)_q/h = (q^h - 1)/(h(q-1))$ is continuous at all $h \in \mathbf{Z}_p - \{0\}$ and its limit as $h \rightarrow 0$ is $\frac{\log_p q}{q-1} \neq 0$ (even if $q = 1$), the function $(h)_q/h$ is continuous and nowhere vanishing. So by Lemma 4.2 (with $h-1$ as the variable), $f'(x)$ exists if and only if $(\Delta_q^m f)(x)/(m)_q \rightarrow 0$ and then $f'(x)$ has the indicated form.

(ii) We consider only suitably small h , say $h = p^N z$ for $z \in \mathbf{Z}_p$. For $z \neq 0$,

$$\frac{f(x+p^N z) - f(x)}{p^N z} = \sum_{m \geq 1} (\Delta_q^m f)(x) \frac{1}{p^N z} \binom{p^N z}{m}_q,$$

and by (3.1),

$$\binom{p^N z}{m}_q = \begin{cases} \binom{z}{m/p^N}, & \text{if } p^N | m; \\ 0, & \text{if } p^N \nmid m, \end{cases}$$

so

$$\begin{aligned} \frac{f(x + p^N z) - f(x)}{p^N z} &= \sum_{l \geq 1} (\Delta_q^{p^N l} f)(x) \frac{1}{p^N z} \binom{z}{l} \\ &= \sum_{l \geq 1} \frac{(\Delta_q^{p^N l} f)(x)}{p^{Nl}} \binom{z-1}{l-1}. \end{aligned}$$

Apply Lemma 4.2 (for $q = 1$) with $z - 1$ as the variable. \square

Let's unify both parts of this theorem. For q not a root of unity, $\frac{\log_p q}{q-1} \frac{1}{(m)_q} = \frac{\log_p q}{q^m - 1}$, while Lemma 4.1(vi) shows that for q a root of unity, $(\log_p q)/(q^m - 1)$ equals $1/m$ when $q^m = 1$ and equals 0 otherwise. Moreover, if q is a root of unity of order p^N , then $\binom{-1}{p^N l - 1}_q = (-1)^{l-1}$ for $l \geq 1$. So for any $|q - 1| < 1$, a root of unity or not, f is differentiable at x if and only if $\lim_{m \rightarrow \infty} (\Delta_q^m f)(x) \frac{\log_p q}{q^m - 1} = 0$, in which case

$$f'(x) = \sum_{m \geq 1} (\Delta_q^m f)(x) \frac{\log_p q}{q^m - 1} \binom{-1}{m-1}_q.$$

In particular,

$$f'(0) = \sum_{m \geq 1} c_{m,q} \frac{\log_p q}{q^m - 1} \binom{-1}{m-1}_q.$$

When $f(x) = \sum c_n \binom{x}{n}$ is differentiable and f' is continuous, Mahler [18, Theorem 4] gives the Mahler expansion for f' :

$$(4.3) \quad f'(x) = \sum_{n \geq 0} \left(\sum_{j \geq 1} \frac{c_{n+j}}{j} (-1)^{j-1} \right) \binom{x}{n}.$$

For the q -analogue, we use the following q -analogue of [22, Prop. 47.4]:

$$p^k \leq n < p^{k+1} \Rightarrow \left| \binom{x}{n} - \binom{y}{n} \right| \leq p^k |x - y|.$$

LEMMA 4.3. Let $n \geq 1$, $p^k \leq n < p^{k+1}$.

(i) When q is not a (nontrivial) root of unity,

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \frac{1}{|(p^k)_q|} |(x)_q - (y)_q| \leq \frac{1}{|(p^k)_q|} \max(|q-1|, 1/p)^{\text{ord}(x-y)}.$$

(ii) When q is a root of unity of order p^N ($N \geq 0$) and $x \equiv y \pmod{p^N}$,

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq p^k |x - y|.$$

Proof. i) Let $x = y + z$, so by Theorem 3.1,

$$\binom{x}{n}_q - \binom{y}{n}_q = \sum_{j=1}^n \frac{(z)_q}{(j)_q} \binom{z-1}{j-1}_q \binom{y}{n-j}_q q^{j(y+j-n)},$$

hence

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \max_{1 \leq j \leq n} \left| \frac{\binom{z}{j}_q}{\binom{j}{j}_q} \right| = \max_{m \leq k} \frac{1}{|(p^m)_q|} |(x)_q - (y)_q|.$$

ii) The difference vanishes if $n < p^N$, so we may assume $n \geq p^N$, i.e., $k \geq N$. Let $x \equiv y \equiv r \pmod{p^N}$, $0 \leq r \leq p^N - 1$. Write $x = p^N x' + r$, $y = p^N y' + r$, $n = p^N l + s$, $0 \leq s \leq p^N - 1$, so $p^{k-N} \leq l < p^{k+1-N}$. Then $\binom{x}{n}_q - \binom{y}{n}_q = \left(\binom{x'}{l} - \binom{y'}{l} \right) \binom{r}{s}_q$, so (knowing the case $q = 1$ already)

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \left| \binom{x'}{l} - \binom{y'}{l} \right| \leq p^{k-N} |x' - y'| = p^k |x - y|.$$

□

If $|q - 1| < (1/p)^{1/(p-1)}$, then part (i) reduces to $|\binom{x}{n}_q - \binom{y}{n}_q| \leq p^k |x - y|$, which (for $q \in \mathbf{Z}$) is a special case of [8, Theorem 4.5].

Here is the q -analogue of the Mahler expansion of f' when f' is continuous, extending (4.3).

THEOREM 4.3. *Let $f(x) = \sum_{n \geq 0} c_{n,q} \binom{x}{n}_q$ be a continuous function from \mathbf{Z}_p to K with a continuous derivative. The q -Mahler expansion of f' is*

$$\begin{aligned} f'(x) &= \sum_{n \geq 0} \left(n c_{n,q} \log_p q + \sum_{j \geq 1} c_{n+j,q} \frac{\log_p q}{q^j - 1} \binom{-1}{j-1}_q q^{-jn} \right) \binom{x}{n}_q \\ &= \sum_{n \geq 0} \left(n c_{n,q} \log_p q + \sum_{j \geq 1} c_{n+j,q} \frac{\log_p q}{q^j - 1} (-1)^{j-1} q^{-j(j-1)/2-jn} \right) \binom{x}{n}_q. \end{aligned}$$

Proof. Apply $\lim_{m \rightarrow \infty} (\Delta_q^m f)(x) \frac{\log_p q}{q^m - 1} = 0$ at $x = 0, 1, 2, \dots$ to see $\lim_{m \rightarrow \infty} c_{n+m,q} \frac{\log_p q}{q^m - 1} = 0$ for all $n \in \mathbf{N}$.

For $y \neq 0$,

$$\frac{f(x+y) - f(x)}{y} = \sum_{n \geq 0} \left(\frac{(\Delta_q^n f)(y) - c_{n,q}}{y} \right) \binom{x}{n}_q.$$

By (4.1),

$$\frac{(\Delta_q^n f)(y) - c_{n,q}}{y} = c_{n,q} \left(\frac{q^{yn} - 1}{y} \right) + \sum_{j \geq 1} c_{n+j,q} q^{n(y-j)} \frac{1}{y} \binom{y}{j}_q.$$

How does each term behave as $y \rightarrow 0$? The first term tends to $c_{n,q} \log_p(q^n) = n c_{n,q} \log_p q$. For the other terms,

$$\begin{aligned} q^{n(y-j)} \frac{1}{y} \binom{y}{j}_q &= q^{n(y-j)} \frac{q^y - 1}{y} \frac{1}{q^j - 1} \binom{y-1}{j-1}_q \\ &\rightarrow \frac{\log_p q}{q^j - 1} \binom{-1}{j-1}_q q^{-jn} \\ &= \frac{\log_p q}{q^j - 1} (-1)^{j-1} q^{-j(j-1)/2-jn}. \end{aligned}$$

This calculation is valid only if $q^j \neq 1$, but the result is true if $q^j = 1$ by using (3.1). So we expect

$$(4.4) \quad f'(x) = \sum_{n \geq 0} \left(n c_{n,q} \log_p q + \sum_{j \geq 1} c_{n+j,q} \frac{\log_p q}{q^j - 1} (-1)^{j-1} q^{-j(j-1)/2 - jn} \right) \binom{x}{n}_q.$$

However, though we know $\lim_{j \rightarrow \infty} c_{n+j,q} \frac{\log_p q}{q^j - 1} = 0$ for each n , so the putative q -Mahler coefficients of f' in (4.4) do make sense, we don't yet know

$$\lim_{n \rightarrow \infty} \sum_{j \geq 1} c_{n+j,q} \frac{\log_p q}{q^j - 1} \binom{-1}{j-1}_q q^{-jn} = 0,$$

so convergence of the infinite series over n in (4.4) is not clear. To get around this, we use the idea of Mahler from his proof of Theorem 4.3 at $q = 1$, namely by the hypothesis of continuity of f' it suffices to verify (4.4) when $x = m \in \mathbf{N}$. In this case the sum over n becomes finite:

$$\frac{f(m+y) - f(m)}{y} = \sum_{n=0}^m \left(c_{n,q} \left(\frac{q^{yn} - 1}{y} \right) + \sum_{j \geq 1} c_{n+j,q} q^{n(y-j)} \frac{1}{y} \binom{y}{j}_q \right) \binom{m}{n}_q.$$

The outer sum is finite, so to verify termwise evaluation of $\lim_{y \rightarrow 0}$ all we need to do is check

$$\lim_{y \rightarrow 0} c_{n+j,q} \frac{1}{y} \binom{y}{j}_q = c_{n+j,q} \frac{\log_p q}{q^j - 1} \binom{-1}{j-1}_q$$

uniformly in j (but perhaps not in q or n).

Case 1: q is a root of unity of order p^N , so $\lim_{j \rightarrow \infty} c_{n+j,q}/j = 0$, as j runs through multiples of p^N .

If $q^j \neq 1$, then $\binom{y}{j}_q = 0$ for $|y| \leq 1/p^N$.

If $q^j = 1$, say $j = p^N j'$, then

$$\lim_{y \rightarrow 0} c_{n+j,q} \frac{1}{y} \binom{y}{j}_q = \lim_{z \rightarrow 0} c_{n+j,q} \frac{1}{j} \binom{z-1}{j'-1}.$$

We consider the difference

$$c_{n+j,q} \frac{1}{j} \binom{z-1}{j'-1} - c_{n+j,q} \frac{1}{j} \binom{-1}{j-1}_q = \frac{c_{n+j,q}}{j} \left(\binom{z-1}{j'-1} - \binom{-1}{j'-1} \right).$$

Choose a power of p , say p^r , such that $|c_{n+j,q}/j| \leq \delta$ for $j \geq p^r$ (and $p^N |j|$). For $j < p^r$, $\binom{z-1}{j'-1} - \binom{-1}{j'-1}$ has size at most $p^{r-1} |z|$ by Lemma 4.3.

Therefore

$$\lim_{y \rightarrow 0} c_{n+j,q} \frac{1}{y} \binom{y}{j}_q = c_{n+j,q} \frac{\log_p q}{q^j - 1} \binom{-1}{j-1}_q,$$

uniformly in j .

Case 2: q is not a root of unity.

So $\log_p q \neq 0$, hence $\lim_{j \rightarrow \infty} \frac{c_{n+j,q}}{q^j - 1} = 0$.

Since

$$\begin{aligned}
c_{n+j,q} \frac{1}{y} \binom{y}{j}_q - c_{n+j,q} \frac{\log_p q}{q^j - 1} \binom{-1}{j-1}_q &= \frac{c_{n+j,q}}{q^j - 1} \left(\frac{q^y - 1}{y} \binom{y-1}{j-1}_q - \log_p q \binom{-1}{j-1}_q \right) \\
&= \frac{c_{n+j,q}}{q^j - 1} \left(\frac{q^y - 1}{y} - \log_p q \right) \binom{y-1}{j-1}_q + \\
&\quad \frac{c_{n+j,q}}{q^j - 1} \log_p q \left(\binom{y-1}{j-1}_q - \binom{-1}{j-1}_q \right),
\end{aligned}$$

we need to show that

$$\lim_{y \rightarrow 0} \frac{c_{n+j,q}}{q^j - 1} \left(\binom{y-1}{j-1}_q - \binom{-1}{j-1}_q \right) = 0$$

uniformly in j . For $\delta > 0$, choose p^r so $|c_{n+j,q}/(q^j - 1)| \leq \delta$ for $j \geq p^r$. For $j < p^r$, Lemma 4.3 implies

$$\left| \binom{y-1}{j-1}_q - \binom{-1}{j-1}_q \right| \leq \frac{1}{|(p^r - 1)_q|} \max(|q - 1|, 1/p)^{\text{ord}(y)},$$

which is $\leq \delta$ for $\text{ord}(y)$ large enough. \square

So for f' continuous and q not a root of unity,

$$f'(x) = \frac{\log_p q}{q - 1} \sum_{n \geq 0} \left((q - 1) n c_{n,q} + \sum_{j \geq 1} \frac{c_{n+j,q}}{\binom{j}{q}} (-1)^{j-1} q^{-j(j-1)/2 - jn} \right) \binom{x}{n}_q,$$

while for q a root of unity of order p^N ,

$$f'(x) = \sum_{n \geq 0} \left(\sum_{\substack{j \geq 1 \\ p^N | j}} \frac{c_{n+j,q}}{j} (-1)^{j/p^N - 1} \right) \binom{x}{n}_q.$$

The Mahler expansion characterizes analyticity: $\sum c_n \binom{x}{n}$ is analytic if and only if $c_n/n! \rightarrow 0$ [22, Theorem 54.4]. For example, the function q^x is an analytic function of x if and only if $|q - 1| < (1/p)^{1/(p-1)}$, in which case its m th Taylor coefficient at $x = 0$ is $(\log_p q)^m/m!$. For other q , $|q^{p^r} - 1| < (1/p)^{1/(p-1)}$ for r large, so $(x)_q$ is locally analytic.

To describe analyticity in terms of q -Mahler expansions, we only consider $|q - 1| < (1/p)^{1/(p-1)}$, since this is the region of q where the functions $\binom{x}{n}_q$ are all analytic. For such q , $|(x)_q| = |x|$. In particular, $|n!| = |(n)_q!|$.

LEMMA 4.4. *Let $a_1, b_1, \dots, a_m, b_m \in K$ with $|a_j|, |b_j| \leq 1$. Then*

$$|a_1 a_2 \cdots a_n - b_1 b_2 \cdots b_n| \leq \max |a_j - b_j|.$$

Proof. In $\mathcal{O}_K/(a_1 - b_1, \dots, a_n - b_n)$, $a_1 \cdots a_n \equiv b_1 \cdots b_n$. \square

THEOREM 4.4. *For $|q - 1| < (1/p)^{1/(p-1)}$, $\sum c_n \binom{x}{n}_q$ is analytic if and only if $c_n/(n)_q! \rightarrow 0$.*

Proof. As with the first proof of Theorem 3.3, we'll get the result for general q from the case $q = 1$ by Lemma 3.2.

Let $A(\mathbf{Z}_p, K) = \{f(x) = \sum a_n x^n : a_n \in K, a_n \rightarrow 0\}$ be the analytic functions from \mathbf{Z}_p to K . It is a K -Banach space under the norm $\|f\| = \max |a_n|$. (This norm does not generally coincide with the sup-norm over \mathbf{Z}_p , e.g., $\|x^p - x\| = 1$, but $|x^p - x|_{\text{sup}} = 1/p$.)

Writing

$$\sum a_n x^n = \sum b_n x(x-1) \cdots (x-n+1) = \sum n! b_n \binom{x}{n},$$

we see $a_n - b_n \in \mathbf{Z}[[b_{n+1}, b_{n+2}, \dots]]$, so $\max |a_n| = \max |b_n|$. Therefore the norm in $A(\mathbf{Z}_p, K)$ of an analytic function written as $\sum c_n \binom{x}{n}$ is $\max |c_n/n!|$. In other words, the functions $n! \binom{x}{n} = x(x-1) \cdots (x-n+1)$ are an orthonormal basis of $A(\mathbf{Z}_p, K)$.

The theorem amounts to showing the functions $(n)_q! \binom{x}{n}_q = (x)_q(x-1)_q \cdots (x-n+1)_q$ are an orthonormal basis of $A(\mathbf{Z}_p, K)$. To show this we compare these functions to $n! \binom{x}{n}$ in order to use Lemma 3.2. By Lemma 4.4, it suffices to find an $\varepsilon < 1$ such that $\|(x-j)_q - (x-j)\| \leq \varepsilon$ for all $j \in \mathbf{N}$. Well,

$$(4.5) \quad (x-j)_q - (x-j) = \left(\frac{\log_p q}{q-1} - 1\right)(x-j) + \frac{\log_p q}{q-1} \sum_{r \geq 2} \frac{(\log_p q)^{r-1}}{r!} (x-j)^r.$$

We want a uniform upper bound < 1 on the Taylor coefficients. (The definition of the norm on $A(\mathbf{Z}_p, K)$ is based on a Taylor expansion around 0, but recentering the series at j does not affect the maximum size of the Taylor coefficients.)

The coefficient of $x-j$ on the right side of (4.5) is

$$\frac{\log_p q}{q-1} - 1 = \sum_{n \geq 2} \frac{(q-1)^{n-1}}{n} (-1)^{n-1}.$$

Note $|(q-1)^{n-1}/n| \leq |(q-1)^{n-1}/n!|$. By Lemma 4.1(iv), the coefficients of the higher powers of $x-j$ in (4.5) have size

$$\left| \frac{\log_p q (\log_p q)^{r-1}}{q-1 r!} \right| = \left| \frac{(q-1)^{r-1}}{r!} \right|.$$

So provided $\sup_{r \geq 2} |(q-1)^{r-1}/r!| < 1$, we're done. Letting $s_p(r)$ be the sum of the base p digits of r ,

$$\left| \frac{(q-1)^{r-1}}{r!} \right| = |q-1|^{r-1} p^{(r-s_p(r))/(p-1)} \leq |q-1|^{r-1} p^{(r-1)/(p-1)} \leq |q-1| p^{1/(p-1)}.$$

□

COROLLARY 4.1. *For $|q-1| < (1/p)^{1/(p-1)}$ and $|t| < 1$, $(1+t)^{(x;q)}$ is analytic on \mathbf{Z}_p if and only if $|t| < (1/p)^{1/(p-1)}$.*

We now connect the work here with that of van Hamme and Verdoort. They consider the following. Let $a, q \in \mathbf{Z}_p^\times$, perhaps $q \not\equiv 1 \pmod p$, and assume q is not a root of unity. Let V_q denote the closure of the set $\{aq^n\}_{n \geq 0}$ in \mathbf{Z}_p . It is a compact subset of \mathbf{Z}_p , and open since q is not a root of unity. As $q \rightarrow 1$, V_q "shrinks" to $\{a\}$. In [23], van Hamme proves every continuous function $f: V_q \rightarrow \mathbf{Q}_p$ has the form

$$(4.6) \quad f(x) = \sum_{n \geq 0} \frac{(D_q^n f)(a)}{(n)_q!} (x-a)^{(n;q)}$$

for $x \in V_q$, where $(D_q f)(x) := (f(qx) - f(x))/(qx - x)$ is the q -derivative, D_q^n its n th iterate. Note that the domain V_q of the function depends on q and a . Having $(n)_q!$ in the denominator of (4.6) keeps q away from roots of unity.

When $q \in 1 + p\mathbf{Z}_p$ and is not a root of unity, (4.6) is essentially a q -Mahler expansion. Indeed, in this case the elements of V_q have the form $x = aq^y$ for unique $y \in \mathbf{Z}_p$, in which case

$$\begin{aligned} \frac{(D_q^n f)(a)}{(n)_q!} (x - a)^{(n;q)} &= \frac{(D_q^n f)(a)}{(n)_q!} (aq^y - a)^{(n;q)} \\ &= (D_q^n f)(a) \cdot a^n (q - 1)^n \frac{(q^y - 1)(q^y - q) \cdots (q^y - q^{n-1})}{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)} \\ &= (D_q^n f)(a) \cdot a^n (q - 1)^n q^{n(n-1)/2} \binom{y}{n}_q. \end{aligned}$$

This last expression has an alternate form by [23, Lemma 3]:

$$(D_q^n f)(a) \cdot a^n (q - 1)^n q^{n(n-1)/2} = \sum_{k=0}^n (-1)^k q^{k(k-1)/2} \binom{n}{k}_q f(aq^{n-k}).$$

This goes back to Jackson [14, Eq. 12].

Letting $g(y) = f(aq^y)$ be the pullback of f to a continuous function on \mathbf{Z}_p , van Hamme's expansion (4.6) becomes

$$g(y) = \sum_{n \geq 0} \left(\sum_{k=0}^n (-1)^k q^{k(k-1)/2} \binom{n}{k}_q g(n-k) \right) \binom{y}{n}_q,$$

which is the q -Mahler expansion of g . But q -Mahler expansions do allow q to be a root of unity, as well as to lie outside of \mathbf{Q}_p , though subject to the restriction $|q - 1| < 1$. In [6], a q -analogue of Mahler expansions will be described for $q \in K, |q| = 1$, that will reduce to van Hamme's expansion when $q \in \mathbf{Z}_p^\times$ and q is not a root of unity.

In [24, Theorem 3], van Hamme gives a remainder formula for the Mahler expansion. For a complete extension field K/\mathbf{Q}_p and a continuous function $f: \mathbf{Z}_p \rightarrow K$ with Mahler coefficients c_n ,

$$(4.7) \quad f(x) = c_0 + c_1 \binom{x}{1} + \cdots + c_n \binom{x}{n} + \Delta^{n+1} f *' \binom{\cdot}{n},$$

where $*'$ is a modified convolution of continuous functions that we now recall. For two continuous functions g and h from \mathbf{Z}_p to K , let $g * h: \mathbf{Z}_p \rightarrow K$ be the p -adic interpolation to \mathbf{Z}_p of the function $\mathbf{N} \rightarrow K$ given by $n \mapsto \sum_{k=0}^n g(k)h(n-k)$. (For a proof that this sequence interpolates, see [22, Exer. 34.E, Exer. 52.J] or [24, Lemma 1].) The operation $*$ is an associative, commutative multiplication on $C(\mathbf{Z}_p, K)$ and $|g * h|_{\text{sup}} \leq |g|_{\text{sup}} |h|_{\text{sup}}$. By definition, $(g *' h)(x) := (g * h)(x - 1)$. Since $\Delta^{n+1} f \rightarrow 0$ in $C(\mathbf{Z}_p, K)$, (4.7) is a Mahler expansion with remainder.

Here is the q -Mahler expansion with remainder.

THEOREM 4.5. *Choose $q \in K$ with $|q - 1| < 1$ and $f \in C(\mathbf{Z}_p, K)$. Letting $c_{0,q}, c_{1,q}, \dots$ be the q -Mahler coefficients of f ,*

$$f(x) = c_{0,q} + c_{1,q} \binom{x}{1}_q + \cdots + c_{n,q} \binom{x}{n}_q + \Delta_q^{n+1} f *' \binom{\cdot}{n}_q,$$

Our proof below will be a translation of Verdoordt's ideas in [25], where she proves a version of this expansion with remainder for functions on the sets V_q . To simplify the comparison with [25], we write the variable in \mathbf{Z}_p as y .

For $y \in \mathbf{Z}_p$, set $\mathcal{U}_n(y) = q^{ny}$, so $\mathcal{U}_0(y) = \binom{y}{0}_q$. (The functions $\mathcal{U}_n = \mathcal{U}_{n,q}$ were already used in Section 3.)

LEMMA 4.5. *For any $n \geq 0$, $f = f(0)\mathcal{U}_n + (E - q^n)f *' \mathcal{U}_n$.*

Proof. We evaluate the right hand side at $y = m \in \mathbf{Z}^+$:

$$\begin{aligned} ((E - q^n)f *' \mathcal{U}_n)(m) &= \sum_{i=0}^{m-1} (f(i+1) - q^n f(i)) q^{n(m-1-i)} \\ &= \sum_{i=0}^{m-1} f(i+1) q^{n(m-(i+1))} - \sum_{i=0}^{m-1} f(i) q^{n(m-i)} \\ &= f(m) - f(0) q^{nm}. \end{aligned}$$

□

LEMMA 4.6. *For all n ,*

$$\mathcal{U}_{n+1} *' \binom{\cdot}{n}_q = \binom{\cdot}{n+1}_q.$$

Proof. Using the first recursion in (2.3),

$$\begin{aligned} \binom{m}{n+1}_q &= \binom{m-1}{n}_q + q^{n+1} \binom{m-1}{n+1}_q \\ &= \binom{m-1}{n}_q + q^{n+1} \binom{m-2}{n}_q + q^{2(n+1)} \binom{m-2}{n+1}_q \\ &= \sum_{i=0}^{m-1} \binom{m-1-i}{n}_q q^{i(n+1)} \\ &= \mathcal{U}_{n+1}(y) * \binom{y}{n}_q \quad \text{at } y = m-1 \\ &= \mathcal{U}_{n+1}(y) *' \binom{y}{n}_q \quad \text{at } y = m. \end{aligned}$$

□

Now we prove Theorem 4.5.

Proof. For $y \in \mathbf{Z}_p$,

$$\begin{aligned} f(y) &= f(0)\mathcal{U}_0 + (E - I)f *' \mathcal{U}_0 \\ &= f(0) + \Delta f *' \mathcal{U}_0 \\ &= f(0) + ((\Delta f)(0)\mathcal{U}_1 + (E - q)\Delta f *' \mathcal{U}_1) *' \mathcal{U}_0 \text{ by Lemma 4.5} \\ &= f(0) + (\Delta f)(0)(\mathcal{U}_1 *' \mathcal{U}_0) + \Delta_q^2 f *' (\mathcal{U}_1 *' \mathcal{U}_0) \\ &= f(0) + (\Delta f)(0) \binom{y}{1}_q + \Delta_q^2 f *' \binom{y}{1}_q \text{ by Lemma 4.6.} \end{aligned}$$

Assuming

$$f(y) = f(0) + (\Delta_q f)(0) \binom{y}{1}_q + \cdots + (\Delta_q^n f)(0) \binom{y}{n}_q + \Delta_q^{n+1} f *' \binom{\cdot}{n}_q,$$

apply Lemma 4.5 at $n + 1$ with the function $\Delta_q^{n+1} f$, and then use Lemma 4.6. \square

It is left to the reader to extend the q -Mahler expansion and some properties of it in this section to the case when K is a complete field of characteristic p or a complete commutative \mathbf{Z}_p -algebra.

In addition to the q -numbers and q -binomial coefficients we have used, the study of quantum groups has focused attention on the q -analogues

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + \frac{1}{q^{n-3}} + \frac{1}{q^{n-1}} = \frac{1}{q^{n-1}} (n)_{q^2},$$

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q = \frac{1}{q^{n(n-1)/2}} (n)_{q^2!},$$

and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{[m]_q [m-1]_q \cdots [m-n+1]_q}{[n]_q!} = \frac{1}{q^{(m-n)n}} \binom{m}{n}_{q^2}.$$

The extra property these have is invariance when q is replaced by $1/q$.

All the properties of $\binom{m}{n}_q$ have analogues for $\begin{bmatrix} m \\ n \end{bmatrix}_q$, such as

$$[-n]_q = -[n]_q, \quad [n]_{1/q} = [n]_q, \quad [mn]_q = [m]_q [n]_{q^m}, \quad \begin{bmatrix} m \\ n \end{bmatrix}_q \in \mathbf{Z}[q, 1/q],$$

$$\begin{bmatrix} -m \\ n \end{bmatrix}_q = (-1)^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q, \quad \begin{bmatrix} m_1 + m_2 \\ k \end{bmatrix}_q = \sum_{i+j=k} \begin{bmatrix} m_1 \\ i \end{bmatrix}_q \begin{bmatrix} m_2 \\ j \end{bmatrix}_q q^{m_2 i - m_1 j}.$$

That $\begin{bmatrix} m \\ n \end{bmatrix}_q$ is related to $\binom{m}{n}_{q^2}$ means there is a different formula for $\begin{bmatrix} m \\ n \end{bmatrix}_\zeta$ in the case when ζ is an odd or even order root of unity.

For $|q-1| < 1$, we get a continuous extension $\begin{bmatrix} x \\ n \end{bmatrix}_q = (1/q^{n(x-n)}) \binom{x}{n}_{q^2}$, and $|\begin{bmatrix} x \\ n \end{bmatrix}_q - \binom{x}{n}| \leq |q-1|$, so the functions $\begin{bmatrix} x \\ n \end{bmatrix}_q$ form an orthonormal basis of $C(\mathbf{Z}_p, K)$.

It is left to the reader to formulate all the results of this paper so far in this context. As an example of some differences, let $\mathcal{E}_q(X) = \sum X^n / [n]_q!$. Then $\mathcal{E}_{1/q}(X) = \mathcal{E}_q(X)$ and $\mathcal{E}_q(X)\mathcal{E}_q(Y)$ equals

$$\sum_{n \geq 0} \frac{1}{[n]_q!} \left(\sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q X^{n-m} Y^m \right) = \sum_{n \geq 0} \frac{(X + Y/q^{n-1})(X + Y/q^{n-3}) \cdots (X + q^{n-1}Y)}{[n]_q!},$$

where powers of q in consecutive terms of the product on the right hand side differ by two.

Set

$$(X + Y)^{[n; q]} := (X + Y/q^{n-1})(X + Y/q^{n-3}) \cdots (X + q^{n-1}Y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q X^{n-k} Y^k,$$

so $\mathcal{E}_q(X)\mathcal{E}_q(Y) = \sum_{n \geq 0} (X+Y)^{[n; q]} / [n]_q!$ and $(X+Y)^{[m+n; q]} = (X+q^n Y)^{[m; q]} (X+Y/q^m)^{[n; q]}$. Note $(X - X)^{[n; q]} \neq 0$ if n is even. In particular, $\mathcal{E}_q(X)\mathcal{E}_q(-X) \neq 1$, and there doesn't seem to be a simple formula for the coefficients of $\mathcal{E}_q(X)^{-1}$. For example,

$$\mathcal{E}_q(X)^{-1} = 1 - X + \frac{q^2 - q + 1}{q^2 + 1} X^2 - \frac{q^6 - 2q^5 + 2q^4 - q^3 + 2q^2 - 2q + 1}{(1 + q^2)(1 + q^2 + q^4)} X^3 + \cdots,$$

and the numerator of the coefficient of X^3 is irreducible in $\mathbf{Z}[q]$.

We define polynomials $\mu_n(q)$ by $\mathcal{E}_q(X)^{-1} = \sum_{n \geq 0} \mu_n(q) X^n / [n]_q!$, using the notation μ by analogy with combinatorial inversion formulas. Then

$$f(x) = \sum_{n \geq 0} C_{n,q} \begin{bmatrix} x \\ n \end{bmatrix}_q \iff C_{n,q} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mu_{n-k}(q) f(k).$$

5. THE p -ADIC q -GAMMA FUNCTION

To illustrate the possibility of using q -Mahler expansions with a family of functions depending continuously on a parameter, we consider Morita's p -adic Gamma function Γ_p and its q -analogue $\Gamma_{p,q}$ as defined by Koblitz.

For a nonnegative integer n , Morita [20] defines

$$\Gamma_p(n+1) := (-1)^{n+1} \prod_{\substack{1 \leq j \leq n \\ (p,j)=1}} j = \frac{(-1)^{n+1} n!}{p^{[n/p]} [n/p]!}$$

for $n \geq 1$ and $\Gamma_p(1) = -1$. Morita's proof that Γ_p is p -adically continuous is based on congruence properties of the sequence $\{\Gamma_p(n+1)\}$. For our treatment here, it is Barsky's proof [1] of the continuity which is of primary interest. Barsky's method is based on the identity

$$(5.1) \quad \sum_{n \geq 0} \frac{(-1)^{n+1} \Gamma_p(n+1)}{n!} X^n = (1 + X + \cdots + X^{p-1}) e^{X^p/p},$$

which implies that the Mahler coefficients $\tau_p(n)$ (say) of the sequence $\Gamma_p(n+1)$ satisfy

$$(5.2) \quad \sum_{n \geq 0} \frac{(-1)^{n+1} \tau_p(n)}{n!} X^n = (1 + X + \cdots + X^{p-1}) e^{X+X^p/p}.$$

Writing $e^{X+X^p/p} = \sum_{n \geq 0} (b_{p,n}/n!) X^n$, estimates of Dwork [17, p. 320] imply $b_{p,n} \rightarrow 0$ p -adically as $n \rightarrow \infty$, so $\tau_p(n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore Γ_p extends continuously from \mathbf{N} to \mathbf{Z}_p .

We recall Dwork's proof that $b_{p,n} \rightarrow 0$. Multiply $\exp(X + X^p/p)$ by the additional terms $\exp(X^{p^j}/p^j)$ for $j \geq 2$ and then remove them:

$$(5.3) \quad e^{X+X^p/p} = \exp\left(\sum_{j \geq 0} \frac{X^{p^j}}{p^j}\right) \prod_{j \geq 2} e^{-X^{p^j}/p^j}.$$

We want to show $\exp(X + X^p/p)$ is in the space of p -adic divided power series $\sum c_n X^n/n!$ where $c_n \rightarrow 0$. Such series form the Leopoldt space. It is a Banach algebra when we norm such series by $\sup |c_n|$. Since $\exp(\sum_{j \geq 0} X^{p^j}/p^j)$ is the Artin–Hasse series, which has \mathbf{Z}_p -coefficients, it is a Leopoldt series. (Any series with bounded coefficients is a Leopoldt series.) By a direct calculation, $\exp(\pm X^{p^j}/p^j)$ is a Leopoldt series and $\rightarrow 1$ in the Leopoldt norm as $j \rightarrow \infty$. So by completeness the right side of (5.3) is a Leopoldt series. Thus $b_{p,n} \rightarrow 0$.

For $|q-1| < 1$, the q -analogue $\Gamma_{p,q}$ of Γ_p is defined by Koblitz [16] by

$$\Gamma_{p,q}(n+1) := (-1)^{n+1} \prod_{\substack{1 \leq j \leq n \\ (p,j)=1}} \frac{q^j - 1}{q - 1} = (-1)^{n+1} \prod_{\substack{1 \leq j \leq n \\ (p,j)=1}} (1 + q + \cdots + q^{j-1})$$

for $n \geq 1$ and $\Gamma_{p,q}(1) = -1$. For fixed q with $0 < |q - 1| < 1$, Koblitz shows that the sequence $\Gamma_{p,q}(n+1)$ p -adically interpolates to \mathbf{Z}_p by comparing $\Gamma_{p,q}$ with Γ_p , whose continuity is already known. There are alternate proofs of the interpolation for $\Gamma_{p,q}$ (cf. [5]), but we would like to have available a proof of the interpolation based on Barsky's method, proceeding as follows.

For any integer j , $(j)_{q_1} \equiv (j)_{q_2} \pmod{q_1 - q_2}$, so $|\Gamma_{p,q_1}(n+1) - \Gamma_{p,q_2}(n+1)| \leq |q_1 - q_2|$. Thus p -adic interpolation of $\Gamma_{p,q}(n+1)$ for general q will follow from that for a dense set of q . So we may suppose q is not a root of unity, making $(n)!_q$ nonzero for all n .

In this case, which we may assume we are in from now on,

$$\Gamma_{p,q}(n+1) = \frac{(-1)^{n+1}(n)!_q}{\prod_{k \leq [n/p]} (pk)_q} = \frac{(-1)^{n+1}(n)!_q}{(p)_q^{[n/p]} ([n/p])!_{q^p}}.$$

Following Barsky, we consider

$$\begin{aligned} \sum_{n \geq 0} \frac{(-1)^{n+1} \Gamma_{p,q}(n+1)}{(n)!_q} X^n &= \sum_{n \geq 0} \frac{1}{(p)_q^{[n/p]} ([n/p])!_{q^p}} X^n \\ &= \sum_{r=0}^{p-1} \sum_{m \geq 0} \frac{1}{(p)_q^m (m)!_{q^p}} X^{pm+r} \\ &= (1 + X + \cdots + X^{p-1}) \sum_{m \geq 0} \frac{(X^p/(p)_q)^m}{(m)!_{q^p}} \\ &= (1 + X + \cdots + X^{p-1}) E_{q^p}(X^p/(p)_q) \end{aligned}$$

Let $\tau_{p,q}(n)$ be the n th q -Mahler coefficient of the sequence $\Gamma_{p,q}(n+1)$. We want to show $\tau_{p,q}(n) \rightarrow 0$ as $n \rightarrow \infty$. Continuing with the above calculations, we obtain

$$\begin{aligned} \sum_{n \geq 0} \frac{(-1)^{n+1} \tau_{p,q}(n)}{(n)!_q} X^n &= (1 + X + \cdots + X^{p-1}) E_q(-X)^{-1} E_{q^p}(X^p/(p)_q) \\ &= (1 + X + \cdots + X^{p-1}) E_{1/q}(X) E_{q^p}(X^p/(p)_q). \end{aligned}$$

Comparing this with (5.2) shows the q -analogue of $e^{X+X^p/p}$ is apparently

$$E_{1/q}(X) E_{q^p}(X^p/(p)_q) = (E_{1/q}(X) E_{1/q}(-X)) \cdot E_q(X) E_{q^p}(X^p/(p)_q).$$

By the q -Mahler theorem, the existence of a p -adic interpolation for $\Gamma_{p,q}(n+1)$ is thus equivalent to the fact that, when we write

$$E_{1/q}(X) E_{q^p}(X^p/(p)_q) = \sum_{n \geq 0} b_{p,q,n} \frac{X^n}{(n)!_q},$$

the sequence $b_{p,q,n}$ tends to 0 as $n \rightarrow \infty$. This suggests looking at a q -Leopoldt space, namely the q -divided power series $\sum c_n X^n / (n)!_q$ where $c_n \rightarrow 0$. By a direct calculation for $j \geq 2$, $E_{q^{pj}}(X^{pj}/(p^j)_q)$ is a unit in the q -Leopoldt space, so carrying out a q -version of Barsky's argument comes down to checking that a q -analogue of the Artin–Hasse series,

$$(5.4) \quad E_{1/q}(X) \prod_{j \geq 1} E_{q^{pj}}(X^{pj}/(p^j)_q),$$

is in the q -Leopoldt space. (Since $E_{1/q}(X)E_{1/q}(-X)$ is a unit in the q -Leopoldt space, we can replace $E_{1/q}(X)$ with $E_q(X)$ in (5.4) without affecting the property of being or not being a q -Leopoldt series.)

Here we are left with a gap, as we do not see how to establish (5.4) is a q -Leopoldt series without referring to the preexisting fact that $\Gamma_{p,q}(n+1)$ interpolates. Is there a method of analyzing (5.4) without using anything about $\Gamma_{p,q}$, and ideally also not relying on the case $q=1$ first? It may be possible to carry out this task more easily when $|q-1| < (1/p)^{1/(p-1)}$, but ultimately there should be an argument valid for $|q-1| < 1$.

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