ON WEIL'S PROOF OF THE BOUND FOR KLOOSTERMAN SUMS

KEITH CONRAD

ABSTRACT. While most proofs of the Weil bound on one-variable Kloosterman sums over finite fields are carried out in all characteristics, the original proof of this bound, by Weil, assumes the characteristic is odd. We show how to make Weil's argument work in even characteristic, for both ordinary Kloosterman sums and sums twisted by a multiplicative character.

1. INTRODUCTION

For any finite field $\mathbf{F}_q$ of characteristic p, and any $a \in \mathbf{F}_q^\times$, the corresponding Kloosterman sum

$$\mathrm{Kl}(q,a) = \sum_{x \in \mathbf{F}_q^\times} e^{\frac{2\pi i}{p}\mathrm{Tr}(x+a/x)},$$

where $\mathrm{Tr} = \mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}$ is the trace from $\mathbf{F}_q$ to $\mathbf{F}_p$, satisfies the Weil bound

$$|\mathrm{Kl}(q,a)| \leq 2\sqrt{q}. \tag{1}$$

While there are several different proofs of (1) in the literature, such as [1], [5], [11, pp. 86–87] and [12], most citations refer to the proof in [12] (or [13, App. V]) by Weil. Characteristic of Weil's proof is his use of Salié's formula

$$\mathrm{Kl}(q,a) = \sum_{y \in \mathbf{F}_q} \eta(y^2 - 4a) e^{\frac{2\pi i}{p}\mathrm{Tr}(y)}, \tag{2}$$

where η is the quadratic character on $\mathbf{F}_q^\times$. There is a quadratic character on $\mathbf{F}_q^\times$ only for odd q, which has led to the impression that Weil's argument does not work in characteristic 2. Indeed, expository accounts of (1) which rely on Weil's proof either ignore the case of characteristic 2 or give a completely different argument in characteristic 2.

The proofs of (1) listed above, other than Weil's, apply to odd and even q. Our purpose is to add Weil's proof to that category. We will show Weil's technique is capable of proving the bound (1), as well as a bound on twisted Kloosterman sums, for any q, despite the lack of quadratic characters on $\mathbf{F}_{2^r}^\times$.

There are at least two reasons to suspect Weil's idea should extend to characteristic 2. First of all, Salié's formula expresses Kloosterman sums in odd characteristic as a sum $\sum b(y) e^{\frac{2\pi i}{p}\mathrm{Tr}(y)}$ with the function $b(y)$ taking values in 0 or $\{\pm 1\}$. Kloosterman sums in characteristic 2 also look like this by their definition. This reason is not as compelling as the next reason we give, but it was the initial motivation.

The second reason to expect an extension of Weil's proof to characteristic 2 comes from the way Weil introduces the quadratic character η into his argument, for odd q. This character appears only as an intermediate device in the composite map

$$(\mathbf{F}_q[T]/(T^2 - c))^\times \to \mathbf{F}_q^\times \to \{\pm 1\},$$

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where the first map is the norm and the second is η. (This is clearer in [13] than in [12].) Here c is a suitable nonzero element of $\mathbf{F}_q$. The overall effect is to give a quadratic character on $(\mathbf{F}_q[T]/(T^2-c))^\times$, and this overall effect can still be achieved for even q since $(\mathbf{F}_q[T]/(T^2-c))^\times$ has even size.

For applications where Kloosterman sums over finite fields arise just over the prime fields $\mathbf{F}_p$, such as the Kloosterman–Selberg zeta function, Weil's proof of (1) for odd prime q is adequate (insofar as sums over finite fields are concerned) since the case $q = 2$ (so $a = 1$) can be checked directly. While the Weil bound for $q = 2$, $a = 1$ does imply the Weil bound for $q = 2^r$, $a = 1$ by the Hasse-Davenport relations for Kloosterman sums [2], when q is a higher power of 2 there are more choices of a available than just $a = 1$. Kloosterman sums for q a higher power of 2 are particularly of interest in coding theory [8].

Rather than refer the reader to Weil's proof in order to see the similarity between our arguments in characteristic 2 and Weil's in odd characteristic, we treat here both odd and even q. Thus there will be a certain amount of overlap with Weil's proof.

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2. The Basic Characters

Pick $c \in \mathbf{F}_q^\times$. Unlike $\mathbf{F}_q^\times$, the group $(\mathbf{F}_q[T]/(T^2-c))^\times$ is even whether q is even or odd. For q odd, let η be the quadratic character on $\mathbf{F}_q^\times$. Define a quadratic character ψ_c on $(\mathbf{F}_q[T]/(T^2-c))^\times$ by

$$\psi_c(f(T)) = \begin{cases} \eta(f(\sqrt{c})f(-\sqrt{c})), & \text{if } q \text{ is odd,} \\ (-1)^{\mathrm{Tr}(f'(\sqrt{c})/f(\sqrt{c}))}, & \text{if } q \text{ is even.} \end{cases}$$

For odd q, $f(\sqrt{c})f(-\sqrt{c})$ lies in $\mathbf{F}_q$ whether or not c is a square. For even q, we are using peculiarities of characteristic 2: $\sqrt{c}$ lies in $\mathbf{F}_q^\times$ and $f(T) \mapsto f'(\sqrt{c})$ is well-defined modulo $T^2 - c$. Concretely, for any q and any linear polynomial $b_0 + b_1T$ in $\mathbf{F}_q[T]$ which is prime to $T^2 - c$,

$$\psi_c(b_0 + b_1T) = \begin{cases} \eta(b_0^2 - cb_1^2), & \text{if } q \text{ is odd,} \\ (-1)^{\mathrm{Tr}(b_1/(b_0+b_1\sqrt{c}))}, & \text{if } q \text{ is even.} \end{cases}$$

For any q, define a character $\psi_0 \colon (\mathbf{F}_q[T]/T^2)^\times \to \mathbf{C}^\times$ by

$$\psi_0(f(T)) = e^{\frac{2\pi i}{p}\mathrm{Tr}(f'(0)/f(0))}.$$

For $b_0 + b_1T$ prime to T^2, $\psi_0(b_0 + b_1T) = e^{\frac{2\pi i}{p}\mathrm{Tr}(b_1/b_0)}$.

Define the character Ψ_c on $(\mathbf{F}_q[T]/T^2(T^2-c))^\times$ by $\Psi_c = \psi_0\psi_c$. We'll show in Theorem 2 that the L-function of Ψ_c has a Kloosterman sum as a coefficient. For odd q, a character used by Weil in [13, p. 320, §11] (denoted there as λ) is similar to Ψ_c except, as we will see shortly, for the places where the characters ramify.

As with Dirichlet characters, we will say a character ψ on $(\mathbf{F}_q[T]/M(T))^\times$ is primitive if it cannot be defined modulo any proper factor of $M(T)$.

Lemma 1. *The character ψ_0 is trivial on $\mathbf{F}_q^\times$ and is primitive.*

Proof. We check the second part. Since $\psi_0(1 + bT) = e^{\frac{2\pi i}{p}\mathrm{Tr}(b)}$ is not equal to 1 for some b, ψ_0 can not be defined modulo T. □

Lemma 2. *For $c \in \mathbf{F}_q^\times$, the character ψ_c is trivial on $\mathbf{F}_q^\times$ and is primitive.*

Proof. We check the second part. If q is odd and $T^2 - c$ is irreducible, then $\mathbf{F}_q[T]/(T^2-c) = \mathbf{F}_{q^2}$ is a field and $\psi_c = \eta \circ \mathrm{N}_{\mathbf{F}_{q^2}/\mathbf{F}_q}$ is nontrivial, so it is primitive. If q is odd and $T^2 - c$ is

reducible, then $(\mathbf{F}_q[T]/(T^2-c))^\times \cong \mathbf{F}_q^\times \times \mathbf{F}_q^\times$ and ψ_c is the product of η on both factors. Thus ψ_c is not trivial on either factor. Finally, if q is even, then $\mathbf{F}_q[T]/(T^2-c) = \mathbf{F}_q[T]/(T-\sqrt{c})^2$, and the argument is the same as in the previous lemma. □

Lemma 3. *For $c \in \mathbf{F}_q^\times$, the character $\Psi_c = \psi_0\psi_c$ on $(\mathbf{F}_q[T]/T^2(T^2-c))^\times$ is trivial on $\mathbf{F}_q^\times$ and is primitive.*

Proof. Use Lemmas 1 and 2. □

The following theorem contains a version, adequate for our needs, of the functional equation and Riemann hypothesis for L-functions. It is stated in a very concrete form, while the proof amounts to describing this L-function in a more conceptual way by unwinding definitions.

In our application to Kloosterman sums, the polynomial $M(T)$ in the following theorem will be $T^2(T^2-c)$ for suitable nonzero c.

Theorem 1. *For nonconstant monic $M(T) \in \mathbf{F}_q[T]$, let $\psi\colon (\mathbf{F}_q[T]/M(T))^\times \to \mathbf{C}^\times$ be a character which is trivial on $\mathbf{F}_q^\times$ and is primitive. Define the L-function of ψ by*

$$L(U,\psi) = \frac{1}{1-U}\prod_{(\pi,M)=1}\frac{1}{1-\psi(\pi)U^{\deg\pi}},$$

where π runs over the monic irreducible polynomials that do not divide $M(T)$.

Then $L(U,\psi)$ is a polynomial in U of degree $d = \deg M - 2$, the functional equation

$$L(U,\psi) = wq^{d/2}U^d L(1/qU,\overline{\psi})$$

holds, with w a constant of absolute value 1, and the reciprocal roots of $L(U,\psi)$ have absolute value $\sqrt{q}$.

Proof. For any monic $M(T)$ in $\mathbf{F}_q[T]$, let Λ_M be a full set of roots of the Carlitz polynomial attached to $M(T)$. For the definition of Carlitz polynomials and a systematic treatment of the field extensions their roots generate, see [2], [6] or [7]. Our use of Carlitz polynomials supplants Weil's use of class field theory in the proof of (1).

The L-function attached to any character ω on $\mathrm{Gal}(\mathbf{F}_q(T,\Lambda_M)/\mathbf{F}_q(T))$ is, by definition,

$$\mathcal{L}(U,\omega) = \prod_{\{v:\omega(I_v)=1\}}\frac{1}{1-\omega(\mathrm{Fr}_v)U^{\deg v}},$$

where the product ranges over the places v of $\mathbf{F}_q(T)$ for which the inertia group I_v lies in the kernel of ω, and Fr_v is a Frobenius element at v.

There is a natural isomorphism between $\mathrm{Gal}(\mathbf{F}_q(T,\Lambda_M)/\mathbf{F}_q(T))$ and $(\mathbf{F}_q[T]/M(T))^\times$, analogous to the isomorphism between $\mathrm{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q})$ and $(\mathbf{Z}/m\mathbf{Z})^\times$, which lets any character ψ on $(\mathbf{F}_q[T]/M(T))^\times$ be interpreted as a Galois character. Under this isomorphism, a character mod M is trivial on $\mathbf{F}_q^\times$ and primitive precisely when its corresponding Galois character has conductor $\mathrm{div}(M)_0$. Any place v on $\mathbf{F}_q(T)$ which corresponds to a monic irreducible π not dividing M has trivial inertia group and Fr_v is identified with π mod M. The inertia group and Frobenius elements at a place corresponding to some π dividing M can be described in a manner similar to the inertia group and Frobenius elements in $\mathrm{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q})$ at a prime p dividing m, but we will not describe this data here. The inertia and decomposition group at ∞ is $\mathbf{F}_q^\times$.

For a nontrivial character ω on $\mathrm{Gal}(\mathbf{F}_q(T,\Lambda_M)/\mathbf{F}_q(T))$, it is known that $\mathcal{L}(U,\omega)$ is a polynomial of degree $d = \deg \mathfrak{f} - 2$, where $\mathfrak{f}$ is the conductor of ω; the functional equation $\mathcal{L}(U,\omega) = wq^{d/2}U^d\mathcal{L}(1/qU,\overline{\omega})$ holds, with w (the root number) a constant of absolute value 1; and the reciprocal roots of $\mathcal{L}(U,\omega)$ have absolute value $\sqrt{q}$ (Riemann hypothesis).

The L-function of ψ in the theorem is exactly the L-function of ψ as a Galois character, with $1/(1-U)$ being the factor at the place ∞. The properties of $L(U,\psi)$ which are stated in the theorem are exactly the properties of $\mathcal{L}(U,\psi)$ indicated in the previous paragraph. □

3. Computing L-functions

By Lemma 3 and Theorem 1, the L-function $L(U,\Psi_c)$ is a polynomial in U of degree $4-2=2$. We now compute the polynomial.

Theorem 2. *For q odd and $c \in \mathbf{F}_q^\times$,*

$$L(U,\Psi_c) = 1 + \eta(-c)\,\mathrm{Kl}(q,1/4c)U + qU^2.$$

For q even and $c \in \mathbf{F}_q^\times$,

$$L(U,\Psi_c) = 1 + \mathrm{Kl}(q,1/c)U + qU^2.$$

Proof. The coefficient of U in $L(U,\Psi_c)$ is $1+\sum \Psi_c(T+x)$, where the sum is taken over x in $\mathbf{F}_q$ such that $x \neq 0$ and $x^2 \neq c$.

For q odd, this sum is, following Weil,

$$\begin{aligned} 1 + \sum_{\substack{x\in\mathbf{F}_q \\ x\neq 0,\pm\sqrt{c}}} \Psi_c(T+x) &= 1 + \sum_{\substack{x\in\mathbf{F}_q \\ x\neq 0,\pm\sqrt{c}}} \eta(x^2-c)e^{\frac{2\pi i}{p}\mathrm{Tr}(1/x)}, \\ &= 1 + \eta(-c) \sum_{x\neq 0,\pm 1/\sqrt{c}} \eta(x^2-1/c)e^{\frac{2\pi i}{p}\mathrm{Tr}(x)} \\ &= \eta(-c) \sum_{x\in\mathbf{F}_q} \eta(x^2-1/c)e^{\frac{2\pi i}{p}\mathrm{Tr}(x)}. \end{aligned}$$

By (2), this equals $\eta(-c)\,\mathrm{Kl}(q,1/4c)$.

For q even, this sum is

$$\begin{aligned} 1 + \sum_{x\neq 0,\sqrt{c}} \Psi_c(T+x) &= 1 + \sum_{x\neq 0,\sqrt{c}} (-1)^{\mathrm{Tr}(1/(x+\sqrt{c})+1/x)} \\ &= \sum_{x\neq 1/\sqrt{c}} (-1)^{\mathrm{Tr}(x/(1+x\sqrt{c})+x)} \\ &= \sum_{y\neq 1} (-1)^{\mathrm{Tr}(y/\sqrt{c}(1+y)+y/\sqrt{c})} \\ &= \sum_{y\neq 1} (-1)^{\mathrm{Tr}(y^2/\sqrt{c}(1+y))} \\ &= \sum_{x\neq 0} (-1)^{\mathrm{Tr}(x/\sqrt{c}+1/\sqrt{c}x)} \\ &= \mathrm{Kl}(q,1/c). \end{aligned}$$

The coefficient of U^2 (which is irrelevant for proving (1)) can be computed similarly, albeit more tediously. Alternatively, the functional equation $L(U,\Psi_c) = wqU^2L(1/qU,\overline{\Psi_c})$ forces the root number w to equal 1 since the coefficient of U is a Kloosterman sum, up to sign, and Kloosterman sums are real and nonzero. Therefore the coefficient of U^2 is $wq = q$. □

Theorem 3. *For $a \in \mathbf{F}_q^\times$, $|\,\mathrm{Kl}(q,a)| \leq 2\sqrt{q}$.*

Proof. Apply, with suitable c, Theorem 2 and the Riemann hypothesis for $L(U,\Psi_c)$. □

The inequality in Theorem 3 is strict, since $\mathrm{Kl}(q,a) = -1$ in $\mathbf{Z}[e^{\frac{2\pi i}{p}}]/(1-e^{\frac{2\pi i}{p}}) \cong \mathbf{F}_p$.

We conclude with a treatment of twisted Kloosterman sums

$$\mathrm{Kl}(q,a,\chi) = \sum_{x \in \mathbf{F}_q^\times} \chi(x) e^{\frac{2\pi i}{p}\mathrm{Tr}(x+a/x)},$$

where χ is a nontrivial multiplicative character on $\mathbf{F}_q$. These sums need not be real, since $\overline{\mathrm{Kl}(q,a,\chi)} = \overline{\chi}(-a)\,\mathrm{Kl}(q,a,\chi)$. The bound $|\mathrm{Kl}(q,a,\chi)| \leq 2\sqrt{q}$ was first proved in the literature for odd q by Chowla [3], following Weil's method. Other methods establish this bound for general q [5, p. 228], [9, p. 211], and we now show Weil's method can be applied to any q as well.

Theorem 4. *For $a \in \mathbf{F}_q^\times$ and χ nontrivial, $|\mathrm{Kl}(q,a,\chi)| \leq 2\sqrt{q}$.*

Proof. When q is odd and $\chi = \eta$ is the quadratic character on $\mathbf{F}_q^\times$, the desired bound follows from the exact formula [10, Eqn. (54), (57)], [14],

$$\mathrm{Kl}(q,a,\eta) = \begin{cases} 0, & \text{if } \eta(a) = -1, \\ 2\cos(\frac{2\pi}{p}\,\mathrm{Tr}(b))G(\eta), & \text{if } \eta(a) = 1, \end{cases}$$

where $b^2 = 4a$ when $\eta(a) = 1$, and $G(\eta) = \sum \eta(x) e^{\frac{2\pi i}{p}\mathrm{Tr}(x)}$ is the Gauss sum of η.

From now on we assume χ is nonquadratic, which is automatic for q even. Let $\Phi_{c,\chi}$ be the character on $(\mathbf{F}_q[T]/T^2(T^2-c))^\times$ given by

$$\Phi_{c,\chi}(f) = \chi(f(\sqrt{c})f(-\sqrt{c}))\overline{\chi}(f(0))^2.$$

The product $\Phi_{c,\chi}\Psi_c$, as a character mod $T^2(T^2-c)$, is trivial on $\mathbf{F}_q^\times$ and is primitive. (Primitivity, left to the reader to check, requires χ be nonquadratic.) Writing the L-function of $\Phi_{c,\chi}\Psi_c$ as $1 + A_1U + A_2U^2$, we have $|A_1| \leq 2\sqrt{q}$ by the Riemann hypothesis. By a calculation similar to that in the proof of Theorem 2,

$$A_1 = \begin{cases} (\chi\eta)(-c)\sum_{x\in\mathbf{F}_q}(\chi\eta)(x^2-1/c)e^{\frac{2\pi i}{p}\mathrm{Tr}(x)}, & \text{if } q \text{ is odd}, \\ \chi(c)\,\mathrm{Kl}(q,1/c,\chi^2), & \text{if } q \text{ is even}. \end{cases} \tag{3}$$

For even q, squaring is an automorphism of $\mathbf{F}_q$, so $\chi(c)\,\mathrm{Kl}(q,1/c^2,\chi)$ is an alternate formula for A_1.

The analogue of (2) for twisted Kloosterman sums, due to Davenport [4, Theorem 5] and rediscovered by Chowla [3], says for *odd* q that

$$\mathrm{Kl}(q,a,\chi) = \frac{G(\eta)\overline{\chi}(4)}{G(\chi\eta)} \sum_{y\in\mathbf{F}_q} (\chi\eta)(y^2-4a)e^{\frac{2\pi i}{p}\mathrm{Tr}(y)}, \tag{4}$$

provided χ is nonquadratic. (This expresses $\mathrm{Kl}(q,a,\chi)$, up to a factor $G(\eta)\overline{\chi}(4)/G(\chi\eta)$ of absolute value 1, as a sum of $e^{\frac{2\pi i}{p}\mathrm{Tr}(y)}$ weighted by coefficients that are ± 1 times values of χ. The analogous formula for $\mathrm{Kl}(q,a,\chi)$ when q is even is simply its definition.) Therefore $|A_1| = |\mathrm{Kl}(q,1/4c,\chi)|$ for q odd and $|A_1| = |\mathrm{Kl}(q,1/c^2,\chi)|$ for q even.

Choosing c suitably, we obtain the desired bound on $|\mathrm{Kl}(q,a,\chi)|$. □

The Weil bound for twisted Kloosterman sums need not be a strict inequality. For example, if χ is a character of order 4 on $\mathbf{F}_{81}^\times$, a computer calculation shows $\mathrm{Kl}(81,-1,\chi) = 18 = 2\sqrt{q}$ and $\mathrm{Kl}(81,a_8,\chi) = -18i$, where a_8 is a certain element of order 8 in $\mathbf{F}_{81}^\times$ which depends on χ. (The i in $-18i$ is the same square root of -1 as in the $e^{2\pi i/p}$ used to define twisted Kloosterman sums.) Assuming these computer calculations are just approximate, they are exact since $\mathrm{Kl}(81,-1,\chi)$ and $i\,\mathrm{Kl}(81,a_8,\chi)$ are in $\mathbf{Z}$ by Galois theory. I thank Timothy Choi for supplying these numerical examples.

For completeness, we write down the L-function of $\Phi_{c,\chi}\Psi_c$ for nonquadratic χ. When q is odd,

$$L(U, \Phi_{c,\chi}\Psi_c) = 1 + \frac{(\chi\eta)(-4c)G(\chi\eta)}{G(\eta)} \cdot \mathrm{Kl}(q, 1/4c, \chi)U + \chi(-4c)\eta(-1)G(\chi\eta)^2U^2,$$

and when q is even,

$$\begin{aligned} L(U, \Phi_{c,\chi}\Psi_c) &= 1 + \chi(c)\,\mathrm{Kl}(q, 1/c^2, \chi)U + qU^2 \\ &= 1 + \chi(c)\,\mathrm{Kl}(q, 1/c, \chi^2)U + qU^2. \end{aligned}$$

In these L-functions, the coefficient of U was computed in the proof of Theorem 4. When the coefficient of U is nonzero, the coefficient of U^2 can be computed from the functional equation, just as in the case of untwisted Kloosterman sums. That the formula for the coefficient of U^2 is valid even if the coefficient of U vanishes is left to the reader to check.

The L-functions used here to bound $\mathrm{Kl}(q, a, \chi)$ are not particularly elegant, especially for odd q, since we do not get $\pm\,\mathrm{Kl}(q, a, \chi)$ as the coefficient of U in the L-function. This lack of elegance is somewhat forced by Weil's method, since (for odd q) it relies on the ad hoc formulas (2) and (4). We did not set out to make Weil's method enlightening in characteristic 2 (it is already not enlightening in odd characteristic), but only to show that in characteristic 2 Weil's method does in fact work.

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Department of Mathematics, UC San Diego, La Jolla, CA 92037-0112
E-mail address: kconrad@math.ucsd.edu