HARDY–LITTLEWOOD CONSTANTS

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Abstract Extending a technique introduced in Golomb's thesis, we look at a Dirichlet series naturally associated to the Hardy–Littlewood conjecture.

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To Solomon Golomb on his 70-th birthday

1. Introduction

The Hardy–Littlewood conjecture (also called the Bateman–Horn conjecture) predicts how often polynomials take prime values. To be precise, choose $f_1(T), \ldots, f_r(T)$ in $\mathbf{Z}[T]$ and set

$$\pi_{f_1,\ldots,f_r}(x) = \#\{n \le x : f_1(n), \ldots, f_r(n) \text{ are all prime}\}. \tag{1}$$

Assume the following three conditions on the f_j's.

(a) The f_j's are irreducible and pairwise coprime in $\mathbf{Q}[T]$.

(b) For no prime p is the product $f(n) = f_1(n) \cdots f_r(n)$ divisible by p for all integers n. That is, the function $f\colon \mathbf{Z} \to \mathbf{Z}/(p)$ is not identically zero for any prime p.

(c) Each f_j has a positive leading coefficient.

For example, there are only finitely many prime pairs n, n^2+2, and this pair fails condition (b) at $p = 3$. We include (c) only for expository simplicity. There is no need for it if negative primes are allowed.

Conjecture 1 (Hardy–Littlewood) *When (a), (b), and (c) hold,*

$$\pi_{f_1,\ldots,f_r}(x) \sim \frac{C(f_1, \ldots, f_r)}{(\deg f_1) \cdots (\deg f_k)} \frac{x}{(\log x)^r}, \tag{2}$$

where

$$C(f_1,\dots,f_r)=\prod_p \frac{1-\omega_f(p)/p}{(1-1/p)^r} \tag{3}$$

and $\omega_f(p)$ *is the number of roots of* $f(T):=f_1(T)\cdots f_r(T)$ *in* $\mathbf{Z}/(p)$.

The product $C(f_1,\dots,f_r)$, which is called the *Hardy–Littlewood constant* associated to $f_1,\dots,f_r$, converges by (a) and (b), although convergence is usually just conditional.

Example 2 For twin primes, $f(T)=T(T+2)$, $\omega_f(2)=1$, and $\omega_f(p)=2$ when $p>2$. Conjecture 1 in this case predicts

$$\begin{aligned}\pi_{T,T+2}(x) &\sim 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\frac{x}{(\log x)^2}\\ &\approx 1.3203236\frac{x}{(\log x)^2}.\end{aligned}$$

Example 3 We consider primes of the form $f(n)=n^2+1$. Here $\omega_f(p)=1+\chi_4(p)$, with χ_4 the nontrivial Dirichlet character mod 4. Conjecture 1 predicts

$$\begin{aligned}\pi_{T^2+1}(x) &\sim \prod_p\left(1-\frac{\chi_4(p)}{p-1}\right)\cdot\frac{1}{2}\frac{x}{\log x}\\ &\approx .6864067\frac{x}{\log x}.\end{aligned}$$

These two examples, and other special cases of Conjecture 1, were first discussed by Hardy and Littlewood [7]. A general statement like Conjecture 1 appeared in a paper of Bateman and Horn [2].

While numerical data and sieve methods both point in the direction of the Hardy–Littlewood conjecture, the only proved case corresponds to one linear polynomial, which is Dirichlet's theorem on primes in arithmetic progression. (The prime number theorem is the special case $f(T)=T$.) In no other case is it even known that $\pi_{f_1,\dots,f_r}(x)\to\infty$ with x.

As Bateman and Horn explained in [2], the right side of (2) can be motivated by probabilistic heuristics related to the distribution of the primes. There is no reason to expect such heuristics can lead to a proof of the conjecture. For instance, the proof of Dirichlet's theorem is based not on probability, but on the behavior of Dirichlet series on the line $\mathrm{Re}(s)=1$.

Our original motivation was to address the correctness of the Hardy–Littlewood constant: if (2) is true with an unknown constant in front of

$x/(\log x)^r$, must it be the one appearing in (2)? (This question is asked at the end of [1].) Such a result would generalize part of Chebyshev's evidence for the prime number theorem: if $\pi(x) \sim Cx/\log x$ for some C, then $C = 1$.

Although the question in the previous paragraph remains open, consideration of it led us to an analytic, rather than probabilistic, heuristic whose validity would imply an analytic form of the Hardy–Littlewood conjecture. In particular, we have a new heuristic explanation for the Hardy–Littlewood constant (see Theorem 27). Our basic idea is an extension of a technique used on twin primes in Golomb's thesis [5], [6].

The paper is organized as follows. In Section 2, we state a slightly broader version of Conjecture 1. In Section 3, we introduce a Dirichlet series $F(s)$ related to the conjecture. Section 4 adapts ideas from Golomb's thesis to our setting. In Section 5 our basic heuristic is described (Assumption 24), and implications suggested by it are proved in Sections 6 and 7. We make some concluding remarks in Section 8. It is worth stressing that all lemmas and theorems in this paper are proved unconditionally.

Our notation is largely standard.

2. Z-Valued Polynomials

We will consider the Hardy–Littlewood conjecture not only for polynomials in $\mathbf{Z}[T]$, but also for polynomials in $\mathbf{Q}[T]$ which send $\mathbf{Z}$ to $\mathbf{Z}$. Such polynomials are the $\mathbf{Z}$-linear combinations of binomial coefficient polynomials $\binom{T}{n}$, such as $\binom{T}{3} + 5\binom{T}{2} + 10$. We recall in this section how the Hardy–Littlewood conjecture is stated in this case.

Let $f_1(T), \ldots, f_r(T)$ be polynomials in $\mathbf{Q}[T]$, with product $f(T)$. Assume $f_j(\mathbf{Z}) \subset \mathbf{Z}$ for all j, so $\pi_{f_1,\ldots,f_r}(x)$, as defined in (1), continues to make sense. Conditions (a), (b), and (c) also continue to be meaningful, but $\omega_f(p)$ may not, since $f\colon \mathbf{Z} \to \mathbf{Z}$ may not give a well-defined function from $\mathbf{Z}/(p)$ to $\mathbf{Z}/(p)$ when f does not have integer coefficients. For example, $\binom{T}{3} + 5\binom{T}{2} + 10$ is not a well-defined function from $\mathbf{Z}/(2)$ to $\mathbf{Z}/(2)$: at $T = 0$ it is 10 and at $T = 2$ it is 15.

The generalization of $\omega_f(p)/p$ (which is more basic than the numerator $\omega_f(p)$) from $f \in \mathbf{Z}[T]$ to the case when $f \in \mathbf{Q}[T]$ with $f(\mathbf{Z}) \subset \mathbf{Z}$ is a p-adic density $\delta_f(p)$, which is defined in any of the following equivalent ways.

1. Viewing f as a function $\mathbf{Z} \to \mathbf{Z}/(p)$, there is a modulus p^k such that f is a well-defined function $\mathbf{Z}/(p^k) \to \mathbf{Z}/(p)$. For such k, set

$$\delta_f(p) = \frac{\#\{a \in \mathbf{Z}/(p^k) : f(a) \equiv 0 \bmod p\}}{p^k}.$$

This is independent of the choice of k. (Note that we count solutions to $f(a) \equiv 0 \bmod p$, *not* $f(a) \equiv 0 \bmod p^k$.)

2. Since $f(\mathbf{Z}) \subset \mathbf{Z}$, we have $f(\mathbf{Z}_p) \subset \mathbf{Z}_p$, where $\mathbf{Z}_p$ is the ring of p-adic integers. Define $\delta_f(p)$ to be the Haar measure of
$$\{x \in \mathbf{Z}_p : f(x) \equiv 0 \bmod p\},$$
where the measure is normalized to give $\mathbf{Z}_p$ measure 1.

3. As $s \to 1^+$,
$$\sum_{\substack{n \geq 1 \\ f(n) \equiv 0 \bmod p}} \frac{1}{n^s} \sim \frac{\delta_f(p)}{s-1}.$$

Our first definition of $\delta_f(p)$ is the one given by Bateman and Horn [2]. The equivalence of the first two definitions follows immediately from the definition of Haar measure on the p-adic integers. The equivalence of the first and the third definitions will be the special case of Theorem 23 below, with d prime and $m = 1$.

Example 4 Let $g(T) = \binom{T}{3} + 5\binom{T}{2} + 10$ and $p = 2$. While we have already seen g is not a well-defined function from $\mathbf{Z}/(2)$ to $\mathbf{Z}/(2)$, g does give a well-defined function $\mathbf{Z}/(4) \to \mathbf{Z}/(2)$, and vanishes at three out of the four classes, so $\delta_g(2) = 3/4$.

Note condition (b) in Conjecture 1 means $\delta_f(p) < 1$ for every p, which implies $\delta_{f_j}(p) < 1$ for every p.

The following more general form of Conjecture 1, which also appeared in the paper of Bateman and Horn, is adapted to the case of f_j in $\mathbf{Q}[T]$.

Conjecture 5 *Let $f_1, \ldots, f_r \in \mathbf{Q}[T]$ satisfy $f_j(\mathbf{Z}) \subset \mathbf{Z}$, with product f, and assume the f_j's satisfy* (a), (b), *and* (c) *in Conjecture* 1. *Then* (2) *holds with*
$$C(f_1, \ldots, f_r) = \prod_p \frac{1 - \delta_f(p)}{(1 - 1/p)^r}. \tag{4}$$

3. The Hardy–Littlewood Dirichlet Series

We fix *for the rest of this paper* a set of polynomials $f_1, \ldots, f_r$ in $\mathbf{Q}[T]$, with product f, satisfying the hypotheses of Conjecture 5.

Definition 6 For $\mathrm{Re}(s) > 1$, set
$$F(s) := \sum_{\substack{n \geq 1 \\ \text{all } f_j(n) \geq 2}} \frac{\Lambda(f_1(n)) \cdots \Lambda(f_r(n))}{n^s}.$$

We remind the reader that the von Mangoldt function $\Lambda(n)$ is defined by $\Lambda(p^k) = \log p$ for a prime power $p^k > 1$ and $\Lambda(n) = 0$ otherwise.

In the context of Conjecture 5, $F(s)$ is a natural Dirichlet series to associate to $f_1, \ldots, f_r$, as was done by Baier [1] when all f_j are in $\mathbf{Z}[T]$. Convergence of $F(s)$ is explained in Theorem 9 below.

Definition 7 Set

$$\psi_f(x) := \sum_{\substack{n \le x \\ \text{all } f_j(n) \ge 2}} \Lambda(f_1(n)) \cdots \Lambda(f_r(n)).$$

By partial summation, convergence of $F(s)$ for $\mathrm{Re}(s) > 1$ follows $\psi_f(x) = O(x)$, an estimate we will prove by an argument of Baier [1] that relies on Siegel's finiteness theorem on integral points.

Lemma 8 *Let $g \in \mathbf{Q}[T]$ be irreducible with a positive leading coefficient and let $b \ge 2$ be an integer. Let $N_b(g, x)$ be the number of positive integers $n \le x$ such that $g(n)$ is a perfect b-th power in $\mathbf{Z}$:*

$$N_b(g, x) = \#\{n \le x : g(n) = y^b \text{ for some } y \in \mathbf{Z}\}.$$

Then for each $\varepsilon > 0$, $N_b(g, x) = O(x^\varepsilon)$ as $x \to \infty$, where the O-constant depends on g and ε, but not on b.

This lemma quantifies the idea that an irreducible rational polynomial rarely has pure power values.

Proof: Let $d = \deg g \ge 1$, and fix a choice of $\varepsilon > 0$.

If $b \ge d/\varepsilon$, then the eventual monotonicity of $g(n)$ implies

$$\begin{aligned} N_b(g, x) &= O(g(x)^{1/b}) \\ &= O(x^{d/b}) \\ &= O(x^\varepsilon), \end{aligned}$$

where the O-constant may depend on g, but not on ε or the choice of $b \ge d/\varepsilon$.

Now we suppose $2 \le b \le d/\varepsilon$. Then either $d \ge 3$ and $2 \le b \le d/\varepsilon$, $d = 2$ and $3 \le b \le d/\varepsilon$, or $d = 2$ and $b = 2$. In the first two cases, either d or b is at least 3. For each such b, $N_b(g, x)$ is bounded as $x \to \infty$, by Siegel's theorem on integral points [8, Exer. D.6]. Since the range of b is restricted by d and ε, we can say $N_b(g, x) = O_{g,\varepsilon}(1)$, where the O-constant does not depend on b.

In the third case, when $d = 2$ and $b = 2$, write $g(x) = a_0 + a_1 x + a_2 x^2$ and let t be a common multiple of the denominators of the a_i's. Setting

$h(x) = t^2 g(x) \in \mathbf{Z}[x]$, $N_2(g,x) \leq N_2(h,x)$, so it suffices to assume g has integer coefficients. Then the estimate $N_2(g,x) = O(\log x)$ follows by facts about units in quadratic number fields, as in [1]. ∎

Theorem 9 *We have $\psi_f(x) = O(x)$. In particular, $F(s)$ converges for* $\mathrm{Re}(s) > 1$.

Proof: We first consider the sum

$$\theta_f(x) := \sum_{\substack{n \leq x \\ \text{all } f_j(n) \text{ prime}}} \log f_1(n) \cdots \log f_r(n).$$

A crude upper bound is $\theta_f(x) = O((\log x)^r \pi_{f_1,\ldots,f_r}(x))$. Sieve methods imply $\pi_{f_1,\ldots,f_r}(x) = O(x/(\log x)^r)$, so $\theta_f(x) = O(x)$.

The estimate $\theta_f(x) = O(x)$ passes to $\psi_f(x)$ since for any $\varepsilon > 0$,

$$\begin{aligned} \psi_f(x) - \theta_f(x) &= O\left((\log x)^r \sum_{j=1}^{r} \sum_{2 \leq b \leq \log_2 f_j(x)} N_b(f_j, x)\right) \\ &= O(x^{\varepsilon}(\log x)^{r+1}) \end{aligned}$$

by Lemma 8. ∎

In the proof of Theorem 9, we only needed Lemma 8 for one choice of $\varepsilon < 1$. We will find it convenient to have Lemma 8 with any $\varepsilon > 0$ in Lemma 18.

The significance of $\psi_f(x)$ and $F(s)$ for Conjecture 5 is contained in the following two standard results.

Theorem 10 *If $F(s)$ extends analytically from the half-plane* $\mathrm{Re}(s) > 1$ *to the line* $\mathrm{Re}(s) = 1$ *except for a simple pole at $s = 1$ with residue C, then $\psi_f(x) \sim Cx$.*

Proof: This follows from the Wiener–Ikehara Tauberian theorem, or, since we already know $\psi_f(x) = O(x)$, from the simpler Tauberian theorem of Newman [9]. ∎

Theorem 11 *The following conditions are equivalent, where $C > 0$:*

(1) *As $x \to \infty$, $\psi_f(x) \sim Cx$.*

(2) *As $x \to \infty$,*

$$\pi_{f_1,\ldots,f_r}(x) \sim \frac{C}{(\deg f_1)\cdots(\deg f_r)} \frac{x}{(\log x)^r}.$$

When these conditions hold,

$$\lim_{s\to 1^+}(s-1)F(s) = C.$$

Proof: Use partial summation. ∎

Comparing Theorems 10 and 11 to the Hardy–Littlewood conjecture, it is morally expected that if either $\mathrm{Res}_{s=1}F(s)$ or $\lim_{s\to 1^+}(s-1)F(s)$ exists, then the value will be the Hardy–Littlewood constant $C(f_1,\dots,f_r)$.

Whether or not $F(s)$ extends meromorphically to the line $\mathrm{Re}(s)=1$ is unknown. However, it is reasonable to believe that *if* $F(s)$ is meromorphic at $s=1$, then it has a simple pole there. Indeed, for $\mathrm{Re}(s)>1$ we have the integral representation

$$F(s) = s\int_1^\infty \frac{\psi_f(x)}{x}\frac{\mathrm{d}x}{x^s}.$$

From Theorem 9, for $s>1$

$$F(s) \ll s\int_1^\infty \frac{\mathrm{d}x}{x^s} = \frac{s}{s-1},$$

so $(s-1)F(s)$ is bounded as $s\to 1^+$. Therefore there should be at worst a simple pole at $s=1$. If $F(s)$ were holomorphic at $s=1$, then Landau's lemma on Dirichlet series with nonnegative coefficients implies the Dirichlet series for $F(s)$ converges slightly to the left of $s=1$, so $\psi_f(x)=O(x^{1-\varepsilon})$ for some $\varepsilon>0$. Then $\pi_{f_1,\dots,f_r}(x)=O(x^{1-\varepsilon})$, which is definitely not expected.

The end of Theorem 11 suggests an analytic version of the Hardy–Littlewood conjecture, namely with the same hypotheses as in Conjecture 5,

$$\lim_{s\to 1^+}(s-1)F(s) = C(f_1,\dots,f_r), \tag{5}$$

or equivalently

$$\lim_{s\to 1^+}\frac{\sum_{f_j(n)\ge 2}\Lambda(f_1(n))\cdots\Lambda(f_r(n))n^{-s}}{\sum_{n\ge 2}\Lambda(n)n^{-s}} = C(f_1,\dots,f_r).$$

These depend only on the behavior of $F(s)$ near $s=1$ rather than on the whole line $\mathrm{Re}(s)=1$.

4. A Modified Dirichlet Series

To develop a heuristic method of analyzing $F(s)$ near $s=1$, we will use an alternate expression for its coefficients, based on the following result of Golomb, which deserves to be more widely known.

Lemma 12 (Golomb) *Let $a_1, \ldots, a_r \geq 2$ be pairwise coprime. Then*

$$\Lambda(a_1)\cdots\Lambda(a_r) = \frac{(-1)^r}{r!} \sum_{\substack{d|a_1\cdots a_r \\ d>0}} \mu(d)(\log d)^r.$$

Proof: When $r = 1$ this is $\Lambda(a) = -\sum_{d|a} \mu(d)\log d$, a standard identity.

Now take $r \geq 2$. We compute

$$\begin{aligned} \sum_{d|a_1\cdots a_r} \mu(d)(\log d)^r &= \sum_{\substack{1\leq j\leq r \\ d_j|a_j}} \mu(d_1)\cdots\mu(d_r)(\log d_1 + \cdots + \log d_r)^r \\ &= \sum_{i_1+\cdots+i_r=r} \binom{r}{i_1, \ldots, i_r} \prod_{j=1}^{r} \sum_{d_j|a_j} \mu(d_j)(\log d_j)^{i_j} \end{aligned}$$

and note any term of the outer sum has some $i_j = 0$ or all $i_j = 1$. If $i_j = 0$, the inner sum vanishes (since $a_j > 1$). Therefore the only term occurs when all $i_j = 1$, and is $r!(-\Lambda(a_1))\cdots(-\Lambda(a_r))$. ■

Applying Lemma 12 to $F(s)$, with $a_j = f_j(n)$, requires the $f_j(n)$ are ≥ 2 and pairwise coprime for all n. What if this fails?

Lemma 13 *There is an $m \in \mathbf{Z}^+$ such that, for any $n \in \mathbf{Z}$, the condition $(f(n), m) = 1$ implies $f_1(n), \ldots, f_r(n)$ are pairwise coprime.*

Proof: Since the f_j's are pairwise coprime in $\mathbf{Q}[T]$, we can write

$$A_{ij}(T)f_i(T) + B_{ij}(T)f_j(T) = c_{ij},$$

where $A_{ij}(T), B_{ij}(T) \in \mathbf{Z}[T]$ and c_{ij} is a nonzero integer. If, for some n, $f_i(n)$ and $f_j(n)$ have a common prime factor p, then $p|c_{ij}$.

Let m be the product of all c_{ij}, so for all n in $\mathbf{Z}$, $f_i(n)$ and $f_j(n)$ are coprime in $\mathbf{Z}[1/m]$ for any i and j. Therefore, if $(f(n), m) = 1$ in $\mathbf{Z}$, $(f_i(n), m) = 1$ in $\mathbf{Z}$ for every i, and thus $(f_i(n), f_j(n)) = 1$ in $\mathbf{Z}$ for $i \neq j$ by checking in $\mathbf{Z}[1/m]$.

More succinctly, pairwise coprimality of the $f_j(T)$'s implies the natural map

$$\mathbf{Q}[T]/(f_1\cdots f_r) \to \mathbf{Q}[T]/(f_1) \times \cdots \times \mathbf{Q}[T]/(f_r)$$

is an isomorphism, which implies the natural map

$$\mathbf{Z}[1/m][T]/(f_1\cdots f_r) \to \mathbf{Z}[1/m][T]/(f_1) \times \cdots \times \mathbf{Z}[1/m][T]/(f_r),$$

for some m, is an isomorphism. Such m satisfy the theorem. ■

Definition 14 For any positive integer m, set

$$F_m(s) := \sum_{\substack{(f(n),m)=1 \\ \text{all } f_j(n)\geq 2}} \frac{\Lambda(f_1(n))\cdots\Lambda(f_r(n))}{n^s}.$$

We will use $F_m(s)$ in place of $F(s)$, for m as in Lemma 13, to allow use of Lemma 12. Before we check that $F_m(s)$ has the same relevant analytic features as $F(s)$, we compare $F(s)$ and $F_m(s)$ in some examples.

Example 15 For $r=1$ and $f_1(T)=T$, $F(s)=-\zeta'(s)/\zeta(s)$. Then

$$F_m(s) = \sum_{(n,m)=1} \frac{\Lambda(n)}{n^s} = \sum_{\substack{p^k \\ (p,m)=1}} \frac{\log p}{p^{ks}} = -\frac{\zeta_m'(s)}{\zeta_m(s)},$$

where $\zeta_m(s)$ is the zeta function with its Euler factors at primes dividing m removed. Therefore $F(s) = F_m(s) + \sum_{p|m}\sum_{k\geq 0}(\log p)p^{-ks}$.

Example 16 For twin primes, n and $n+2$ are coprime when both are odd (prime to $m=2$). Then $F(s)=\sum_{n\geq 2}\Lambda(n)\Lambda(n+2)n^{-s}$ and

$$F(s) = F_2(s) + \frac{(\log 2)^2}{2^s}.$$

Example 17 The polynomial values $f_1(n)=5n+3$ and $f_2(n)=6n+5$ are coprime when they are prime to $m=7$. For no n are $5n+3$ and $6n+5$ both powers of 7, so $F(s)=F_7(s)$.

Lemma 18 *For any $m\geq 1$, the Dirichlet series $F(s)-F_m(s)$ converges absolutely for* $\mathrm{Re}(s)>0$.

Proof: For each n contributing a nonzero term to the series

$$F(s)-F_m(s) = \sum_{\substack{(f(n),m)>1 \\ \text{all } f_j(n)\geq 2}} \frac{\Lambda(f_1(n))\cdots\Lambda(f_r(n))}{n^s},$$

some $f_j(n)$ is q^k, where q will be a typical prime factor of m.

Each f_j takes any value at most $\deg f_j$ times, so

$$F(s)-F_m(s) = \text{ finite series } + \sum_{n\in A} \frac{\Lambda(f_1(n))\cdots\Lambda(f_r(n))}{n^s},$$

where

$$A = \{n: \text{ all } f_j(n)\geq 2, \text{ some } f_j(n)=q^b, b\geq 2\}.$$

We now show the partial sums of the coefficients of $F(s) - F_m(s)$ satisfy

$$\sum_{\substack{n\in A\\ n\le x}} \Lambda(f_1(n))\cdots\Lambda(f_r(n)) = O(x^\varepsilon(\log x)^{r+1}) \tag{6}$$

for all $\varepsilon > 0$, where the O-constant may depend on our fixed f. This will prove $F(s) - F_m(s)$ converges for $\mathrm{Re}(s) > 0$.

If, for some $n \le x$, $f_j(n)$ has the form q^b, where q is a prime factor of m and $b \ge 2$, then $b \le K \log x$, where the constant K depends on f_j. Then $\sum_{n\le x, n\in A} \Lambda(f_1(n))\cdots\Lambda(f_r(n))$ grows no faster than

$$(\log x)^r \sum_{2\le b\le K\log x} (N_b(f_1,x) + \cdots + N_b(f_r,x)),$$

where N_b comes from Lemma 8. By Lemma 8, (6) falls out. ∎

By Example 15, the half-plane in Lemma 18 is optimal in general. Theorem 9 and Lemma 18 now give

Theorem 19 *For any m, $F_m(s)$ converges for* $\mathrm{Re}(s) > 1$, *and*

$$\lim_{s\to 1^+}(s-1)F(s) = \lim_{s\to 1^+}(s-1)F_m(s)$$

if either limit exists.

5. The Analytic Heuristic

When m fits the conditions of Lemma 13, then Lemma 12 lets us express $F_m(s)$ as a series of partial zeta functions for $\mathrm{Re}(s) > 1$:

$$\begin{aligned} F_m(s) &= \frac{(-1)^r}{r!} \sum_{\substack{(f(n),m)=1\\ \text{all } f_j(n)\ge 2}} \frac{\sum_{d\mid f(n)} \mu(d)(\log d)^r}{n^s} \\ &= \frac{(-1)^r}{r!} \sum_{\substack{d\ge 1\\ (d,m)=1}} \mu(d)(\log d)^r \left(\sum_{\substack{f(n)\equiv 0 \bmod d\\ (f(n),m)=1}} \frac{1}{n^s} \right), \end{aligned} \tag{7}$$

where the inner sum over n includes the constraint that all $f_j(n) \ge 2$. [1] Aside from this constraint, the other conditions in the inner sum in (7) can be expressed as congruences on n, which will let us use

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[1] What we are looking at here are certain Dirichlet series $h(s) = \sum a_n n^{-s}$ with $a_n \ge 0$ which admit auxiliary expressions $h(s) = \pm\sum \mu(d)c_d g_d(s)$ where $c_d \ge 0$ and $g_d(s) = \sum_{n\in A_d} n^{-s}$ is a partial zeta function running over a subset $A_d \subset \mathbf{Z}^+$; the subsets A_d for different d are allowed to overlap. It is not clear whether such auxiliary expressions, at least for certain A_d, have interesting general features.

Lemma 20 *Let N be a positive integer and a be any integer. The series*

$$\sum_{\substack{n\geq 1 \\ n\equiv a \bmod N}} \frac{1}{n^s},$$

which converges for $\mathrm{Re}(s) > 1$, extends to an entire function except for a simple pole at $s = 1$, with residue $1/N$. Therefore, if the sum runs more generally over positive integers lying in a subset $S \subset \mathbf{Z}/(N)$, the residue at $s = 1$ is $(\#S)/N$.

Proof: This series is N^{-s} times a Hurwitz zeta function. ∎

Theorem 21 *For any integers d and m, the series*

$$\sum_{\substack{f(n)\equiv 0 \bmod d \\ (f(n),m)=1}} \frac{1}{n^s}$$

either extends to an entire function except for a simple pole at $s = 1$ or is identically 0.

The series is identically 0 when there are no n satisfying the condition in the sum, such as $f(T) = T^2 + 1$ and $d = 3$.

Proof: Viewing f as a function $\mathbf{Z} \to \mathbf{Z}/(d)$, choose a multiple D of d so that $f\colon \mathbf{Z}/(D) \to \mathbf{Z}/(d)$ is well-defined. Suppose the zeros of this function in $\mathbf{Z}/(D)$ are $a_1, \ldots, a_k$.

Similarly choose a multiple M of m so $f\colon \mathbf{Z}/(M) \to \mathbf{Z}/(m)$ is well-defined. Since $(f(n), m)$ only depends on n as an element of $\mathbf{Z}/(M)$, we can select congruence class representatives $b_1, \ldots, b_\ell$ in $\mathbf{Z}/(M)$ such that

$$(f(n), m) = 1 \Longleftrightarrow n \equiv b_1, \ldots, b_\ell \bmod M.$$

The series in the theorem takes the form

$$\sum_{\substack{f(n)\equiv 0 \bmod d \\ (f(n),m)=1}} \frac{1}{n^s} = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \sum_{\substack{n\equiv a_i \bmod D \\ n\equiv b_j \bmod M}} \frac{1}{n^s}. \tag{8}$$

Writing each pair of compatible congruence conditions modulo D and M as a set of congruence conditions modulo the least common multiple of D and M, Lemma 20 says the right side of (8) is entire except for a simple pole at $s = 1$, since the residue at $s = 1$ of each inner series is positive. If no pair of congruences modulo D and M is compatible, the series is identically 0. ∎

To compute the residue in Theorem 21, it will be convenient to extend the densities $\delta_f(p)$ from prime arguments to any positive integer.

Definition 22 For any $n \geq 1$, choose a multiple N such that the function $f\colon \mathbf{Z}/(N) \to \mathbf{Z}/(n)$ is well-defined. Set

$$\delta_f(n) := \frac{\#\{a \bmod N : f(a) \equiv 0 \bmod n\}}{N}.$$

The fraction $\delta_f(n)$ does not depend on the choice of N. (When $f(T)$ is in $\mathbf{Z}[T]$, we can take $N = n$.) Easily $\delta_f(n_1 n_2) = \delta_f(n_1)\delta_f(n_2)$ when n_1 and n_2 are relatively prime. By condition (b) at the start of the paper, we do not have $f(\mathbf{Z}) \subset p\mathbf{Z}$ for any prime p, so $\delta_f(n) < 1$ for all $n > 1$. The next theorem shows how values of δ_f arise in a residue.

Theorem 23 *When $(d, m) = 1$ in Theorem 21, the residue of*

$$\sum_{\substack{f(n)\equiv 0 \bmod d \\ (f(n),m)=1}} \frac{1}{n^s}$$

at $s = 1$ is

$$\delta_f(d) \prod_{p|m} (1 - \delta_f(p)).$$

Proof: We can choose D and M so that f gives well-defined functions $\mathbf{Z}/(D) \to \mathbf{Z}/(d)$ and $\mathbf{Z}/(M) \to \mathbf{Z}/(m)$ with D divisible only by prime factors of d and M divisible only by prime factors of m. Therefore $(D, M) = 1$. Each pair of congruence conditions on the right side of (8) can be uniquely solved in $\mathbf{Z}/(DM)$, so the residue at $s = 1$ is

$$\sum_{i=1}^{k} \sum_{j=1}^{\ell} \frac{1}{DM} = \frac{k\ell}{DM}.$$

Then

$$\frac{k}{D} = \frac{\#\{a \bmod D : f(a) \equiv 0 \bmod d\}}{D} = \delta_f(d).$$

Since $\ell = \#\{a \bmod M : f(a) \not\equiv 0 \bmod p \text{ for any } p|m\}$, the ratio ℓ/M equals $\prod_{p|m}(1 - \delta_f(p))$. ∎

The condition $(d, m) = 1$ in Theorem 23 fits the intended application to (7), since the constraint in (7) that all $f_j(n) \geq 2$ has no bearing on the residue.

Now we introduce our basic analytic assumption in the paper.

Assumption 24 With m satisfying Lemma 13, $\lim_{s\to 1^+}(s-1)F_m(s)$ exists and can be computed termwise using (7).

The reasonableness of this assumption comes from the following two examples, where heuristic termwise limits can be rigorously computed in a second way.

Example 25 Recall $\zeta_m(s)$ from Example 15. Since

$$\frac{\zeta_m'(s)}{\zeta_m(s)} = \sum_{\substack{d\geq 1\\(d,m)=1}} \mu(d)\log d \left(\sum_{\substack{n\equiv 0 \bmod d\\(n,m)=1}} \frac{1}{n^s}\right), \tag{9}$$

Assumption 24 suggests that

$$\lim_{s\to 1^+}(s-1)\frac{\zeta_m'(s)}{\zeta_m(s)} \stackrel{?}{=} \sum_{\substack{d\geq 1\\(d,m)=1}} \frac{\mu(d)\log d}{d}\cdot\prod_{p|m}\left(1-\frac{1}{p}\right), \tag{10}$$

where the question mark indicates an equation whose validity remains to be established. The series $\sum_{(d,m)=1}\mu(d)(\log d)/d$ is, at first formally, the Dirichlet series at $s=1$ for

$$-\left(\frac{1}{\zeta_m(s)}\right)' = \frac{\zeta_m'(s)}{\zeta_m(s)^2} = \sum_{\substack{d\geq 1\\(d,m)=1}} \frac{\mu(d)\log d}{d^s}.$$

This equality is true: $(1/\zeta_m(s))'|_{s=1} = \prod_{p|m}(1-1/p)^{-1}$, and it is known that $-\sum_{(d,m)=1}\mu(d)(\log d)/d$ does converge and equal this product. This implies the right side of (10) equals -1. Since $\zeta_m(s)$ has a simple pole at $s=1$, $\lim_{s\to 1^+}(s-1)\zeta_m'(s)/\zeta_m(s)=-1$, so (10) is true and thus Assumption 24 leads to a correct result in this case.

For comparison, writing $-\zeta_m'(s)/\zeta_m(s)$ as $\sum_{(n,m)=1}\Lambda(n)n^{-s}$, which unlike (9) is a series with nonnegative coefficients, the application of Assumption 24 breaks down. The series expression in (9) more closely resembles the intended application (7) than does $\sum_{(n,m)=1}\Lambda(n)n^{-s}$.

Example 26 A calculation analogous to Example 25 holds for the logarithmic derivative of the zeta function of any number field. For an example with a slightly different flavor, pick a number field K and consider the identity

$$\zeta_K(s) = 1 - \sum_{\mathfrak{d}\neq(1)} \mu(\mathfrak{d})\left(\sum_{\mathfrak{a}\equiv 0 \bmod \mathfrak{d}} \frac{1}{\mathrm{N}\mathfrak{a}^s}\right), \tag{11}$$

valid for $\mathrm{Re}(s) > 1$. Here we order the series over ideals $\mathfrak{d}$ according to increasing values of $\mathrm{N}\mathfrak{d}$. Applying Assumption 24 suggests that

$$\lim_{s\to 1^+}(s-1)\zeta_K(s) \stackrel{?}{=} -\sum_{\mathfrak{d}\neq(1)} \frac{\mu(\mathfrak{d})}{\mathrm{N}\mathfrak{d}}\mathrm{Res}_{s=1}\zeta_K(s). \tag{12}$$

Since $\sum_{\mathfrak{d}} \mu(\mathfrak{d})/\mathrm{N}\mathfrak{d} = 0$ (sum over all $\mathfrak{d}$, including $\mathfrak{d} = (1)$), the right side of (12) equals $\mathrm{Res}_{s=1}\zeta_K(s)$, so Assumption 24 leads to a correct result.

Returning to (7), Theorem 23 and Assumption 24 imply

$$\lim_{s\to 1^+}(s-1)F_m(s) \stackrel{?}{=} \frac{(-1)^r}{r!}\prod_{p|m}(1-\delta_f(p)) \sum_{\substack{d\geq 1\\(d,m)=1}} \mu(d)\delta_f(d)(\log d)^r. \tag{13}$$

This suggests the series

$$\sum_{\substack{d\geq 1\\(d,m)=1}} \mu(d)\delta_f(d)(\log d)^r \tag{14}$$

converges, and the Hardy–Littlewood conjecture then suggests the right side of (13) should equal the Hardy–Littlewood constant (4). Unlike the tentative equalities in Examples 25 and 26, (13) can not be made rigorous at present, since we do not have an independent way of computing $\lim_{s\to 1^+}(s-1)F_m(s)$.

In the next two sections we will prove what has been suggested, and this is our main result:

Theorem 27 *For any integer $m \geq 1$, the series* (14) *converges and*

$$\frac{(-1)^r}{r!}\prod_{p|m}(1-\delta_f(p)) \sum_{\substack{d\geq 1\\(d,m)=1}} \mu(d)\delta_f(d)(\log d)^r = C(f_1,\ldots,f_r).$$

In particular, Theorem 19, Assumption 24, and Theorem 27 imply the analytic form of the Hardy–Littlewood conjecture in (5). Of course, this conclusion is conditional since Assumption 24 remains unproved.

It is worth stressing that the proof of Theorem 27 will not rely on any unproved hypotheses (and does not need the restriction on m in Assumption 24). Assumption 24 and the Hardy–Littlewood conjecture are catalysts for bringing (14) to our attention, and for making this series interesting at all.

Example 28 In the case of twin primes, $f(T) = T(T+2)$, $\delta_f(2) = 1/2$, and $\delta_f(p) = 2/p$ for $p > 2$. Taking $m = 2$, Theorem 27 says

$$\frac{1}{4}\sum_{d \text{ odd}} \frac{\mu(d)2^{\nu(d)}(\log d)^2}{d} = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right),$$

where $\nu(d)$ is the number of prime factors of d (that is, $\delta_f(d) = 2^{\nu(d)}/d$ for odd squarefree d). This equality, which was proved by Golomb [5] subject to a weak form of the Riemann hypothesis, is unconditionally true.

6. Proof of Theorem 27

While Theorem 27 was motivated by an unproved assumption about a Dirichlet series, we will prove this theorem by a different method, using an Euler product (Definition 30 below).

Definition 29 For $f(T)$ as in Conjecture 5, define $\omega_f(n)$ by the equation $\delta_f(n) = \omega_f(n)/n$.

Since $\delta_f(n)$ is multiplicative in n, so is $\omega_f(n)$. When $f(T) \in \mathbf{Z}[T]$, $\omega_f(n)$ equals the number of zeros of f in $\mathbf{Z}/(n)$. This interpretation of $\omega_f(n)$ is not always valid for f in $\mathbf{Q}[T]$ with $f(\mathbf{Z}) \subset \mathbf{Z}$, but it is when $n = p$ is a prime and f has p-integral coefficients. Since $\omega_f(p) \leq \deg f$ for these p, the sequence $\omega_f(p)$ (as p runs over all primes) is bounded.

Fix once and for all a positive integer m, without restrictions on it (just as in Theorem 27).

Definition 30 For $\mathrm{Re}(s) > 1$, set

$$G(s) := \prod_{(p,m)=1} \left(1 - \frac{\omega_f(p)}{p^s}\right) = \sum_{(n,m)=1} \frac{\mu(n)\omega_f(n)}{n^s}.$$

Since $\omega_f(p) \leq \deg f$ for all large p and $\omega_f(p) < p$ for all p, $G(s)$ is nonvanishing. We suppress the dependence of $G(s)$ on m and f for simplicity of notation.

Remark 31 When $f \in \mathbf{Z}[T]$ and $m = 1$, the function $G(s)$ has been considered by Kurokawa [10]. He appeals to his elaborate theory of Euler products, which we do not need.

The Dirichlet series for $G(s)$ is supported on squarefree n. Taking derivatives,

$$G^{(j)}(s) = \sum_{(n,m)=1} \frac{\mu(n)\omega_f(n)(-\log n)^j}{n^s} \tag{15}$$

for $\mathrm{Re}(s) > 1$.

A special case of $G(s)$ is $1/\zeta_m(s) = \prod_{(p,m)=1}(1 - p^{-s})$. This is known to extend analytically to the line $\mathrm{Re}(s) = 1$, where it is nonvanishing except for a simple zero (not pole!) at $s = 1$. Moreover, $1/\zeta_m(s)$ and all of its higher derivatives are known to be represented by their Dirichlet series on the line $\mathrm{Re}(s) = 1$.

Theorem 32 *The function $G(s)$ in Definition 30 has an analytic continuation to the line $\mathrm{Re}(s) = 1$, where it is nonvanishing except for an r-th order zero at $s = 1$, where r is the number of irreducible factors of f. Equation (15) is valid on the line $\mathrm{Re}(s) = 1$. In particular,*

$$\begin{aligned} G^{(r)}(1) &= (-1)^r \sum_{(n,m)=1} \frac{\mu(n)\omega_f(n)(\log n)^r}{n} \\ &= (-1)^r \sum_{(n,m)=1} \mu(n)\delta_f(n)(\log n)^r. \end{aligned}$$

The proof of Theorem 32 is deferred to the next section to keep the ideas in the proof of our main result, Theorem 27, as clear as possible.

To prove Theorem 27, we consider the product

$$\zeta(s)^r G(s) = \prod_{p|m} \frac{1}{(1-p^{-s})^r} \prod_{(p,m)=1} \frac{1-\omega_f(p)p^{-s}}{(1-p^{-s})^r}. \tag{16}$$

This is holomorphic and nonvanishing at $s = 1$, by Theorem 32. The value of the left side at $s = 1$ is $G^{(r)}(1)/r!$. Formally substituting $s = 1$ on the right side will be justified in a moment. Making that substitution, we obtain (recalling Definition 29)

$$\frac{G^{(r)}(1)}{r!} = \prod_{p|m} \frac{1}{(1-1/p)^r} \prod_{(p,m)=1} \frac{1-\delta_f(p)}{(1-1/p)^r},$$

which implies by Theorem 32 that the number

$$\frac{(-1)^r}{r!} \prod_{p|m}(1-\delta_f(p)) \sum_{(n,m)=1} \mu(n)\delta_f(n)(\log n)^r$$

equals

$$\begin{aligned} \prod_{p|m}(1-\delta_f(p)) \frac{G^{(r)}(1)}{r!} &= \prod_{p|m} \frac{1-\delta_f(p)}{(1-1/p)^r} \prod_{(p,m)=1} \frac{1-\delta_f(p)}{(1-1/p)^r} \\ &= C(f_1, \ldots, f_r), \end{aligned}$$

and this proves Theorem 27, subject to a proof of Theorem 32 and a proof of

Theorem 33 *The analytic continuation of the right side of* (16) *to* $s = 1$ *has value given by a formal substitution at* $s = 1$.

Proof: It suffices to focus on the product over p prime to m, and check its value at $s = 1$ is obtained by formal substitution. This Euler product $\prod_{(p,m)=1}(1 - \omega_f(p)p^{-s})(1 - p^{-s})^{-r}$ can be written as

$$\prod_{(p,m)=1} \frac{1 - \omega_f(p)p^{-s}}{(1 - \omega_{f_1}(p)p^{-s}) \cdots (1 - \omega_{f_r}(p)p^{-s})} \cdot \prod_{j=1}^{r} (G_j(s)\zeta_m(s)), \tag{17}$$

where $G_j(s)$ is defined in the same way as $G(s)$, using $\omega_{f_j}(p)$ in place of $\omega_f(p)$. (Recall $f = f_1 \cdots f_r$ and $\zeta_m(s) = \prod_{(p,m)=1}(1 - p^{-s})^{-1}$.)

For large p,

$$\omega_f(p) = \omega_{f_1}(p) + \cdots + \omega_{f_r}(p) \tag{18}$$

and $\omega_f(p) \leq \deg f$. Therefore the p-th Euler factor in the first product in (17) looks like $1 + O(p^{-2s})$ for large p, which means the first product and its reciprocal are absolutely convergent on $\mathrm{Re}(s) > 1/2$ when a finite number of initial Euler factors are taken away, in the sense that their formal logarithm series, indexed by prime powers, are absolutely convergent when $\mathrm{Re}(s) > 1/2$. Since $\omega_f(p) < p$ and $\omega_{f_j}(p) < p$ for all p, those initial Euler factors are each holomorphic and nonvanishing on some half-plane $\mathrm{Re}(s) > 1 - \varepsilon$. Therefore the first product in (17) is holomorphic at $s = 1$ with its value given by formal substitution.

The behavior of $G_j(s)\zeta_m(s)$ at $s = 1$ is handled by the next theorem, using $g = f_j$. ■

Theorem 34 *Let* g *be irreducible in* $\mathbf{Q}[T]$ *with* $g(\mathbf{Z}) \subset \mathbf{Z}$ *and* $\delta_g(n) < 1$ *for all* n.

Fix $m \geq 1$, *and set* $G_g(s) := \prod_{(p,m)=1}(1 - \omega_g(p)p^{-s})$. *Then*

$$G_g(s)\zeta_m(s) = \prod_{(p,m)=1} \frac{1 - \omega_g(p)p^{-s}}{1 - p^{-s}},$$

which converges absolutely for $\mathrm{Re}(s) > 1$, *is holomorphic and nonvanishing at* $s = 1$, *and its value at* $s = 1$ *is*

$$\prod_{(p,m)=1} \frac{1 - \delta_g(p)}{1 - 1/p}.$$

Proof: Let $H(s) = G_g(s)\zeta_m(s)$. By Theorem 32 with g in place of f and $r = 1$, $H(s)$ is holomorphic and nonvanishing at $s = 1$.

To compute $H(1)$, we write

$$H(s) = G_g(s)\zeta_K(s) \cdot \frac{\zeta_m(s)}{\zeta_K(s)}, \tag{19}$$

where $K = \mathbf{Q}(\gamma)$ for γ a root of g and $\zeta_K(s)$ is the zeta function of K. Let $Q_p(T)$ be the polynomial defining the p-th Euler factor in $\zeta_K(s)$:

$$\zeta_K(s) := \prod_p \frac{1}{Q_p(p^{-s})}.$$

For all but finitely many p, $\omega_g(p)$ is the number of degree 1 primes lying over p in the ring of integers of K. There is no harm in changing m so $\omega_g(p)$ is described as in the previous sentence when $(p,m) = 1$. Then $Q_p(p^{-s}) = 1 - \omega_g(p)p^{-s} + O(p^{-2s})$ as $s \to 1^+$ for $(p,m) = 1$, which makes

$$\begin{aligned} G_g(s)\zeta_K(s) &= \prod_{(p,m)=1} \frac{1-\omega_g(p)p^{-s}}{Q_p(p^{-s})} \cdot \prod_{p|m} \frac{1}{Q_p(p^{-s})} \\ &= \prod_{(p,m)=1} \left(1 + O\left(\frac{1}{p^{2s}}\right)\right) \cdot \prod_{p|m} \frac{1}{Q_p(p^{-s})}. \end{aligned}$$

This shows we can set $s = 1$ in (19) to obtain

$$H(1) = \prod_{(p,m)=1} \frac{1-\delta_g(p)}{Q_p(1/p)} \cdot \prod_{p|m} \frac{1}{Q_p(1/p)} \cdot \frac{\mathrm{Res}_{s=1}\zeta_m(s)}{\mathrm{Res}_{s=1}\zeta_K(s)}.$$

By Mertens' asymptotic,

$$\prod_{\substack{(p,m)=1 \\ p \le x}} \frac{1}{1-1/p} \sim (\mathrm{Res}_{s=1}\zeta_m(s)) e^{\gamma} \log x,$$

and similarly

$$\prod_{p \le x} \frac{1}{Q_p(1/p)} \sim (\mathrm{Res}_{s=1}\zeta_K(s)) e^{\gamma} \log x.$$

Therefore

$$H(1) = \prod_{(p,m)=1} \frac{1-\delta_g(p)}{Q_p(1/p)} \cdot \prod_{p|m} \frac{1}{Q_p(1/p)} \cdot \prod_{(p,m)=1} \frac{Q_p(1/p)}{1-1/p} \cdot \prod_{p|m} Q_p(1/p),$$

and the desired result falls out. ∎

Remark 35 It is appealing to regard the last part of Theorem 34 as a multiplicative version of Abel's theorem, but there is no Abel's theorem for Euler products (counterexamples are known).

7. Proof of Theorem 32

We split the proof of Theorem 32 into two parts: analytic continuation to $\mathrm{Re}(s) = 1$ and evaluation of its derivatives on this line.

Let γ_j be a root of f_j and set $K_j = \mathbf{Q}(\gamma_j)$. For all but finitely many p, $\omega_{f_j}(p)$ is the coefficient of p^{-s} in $\zeta_{K_j}(s)$, so the p-th factor in the Euler product for

$$\widetilde{G}(s) := G(s)\zeta_{K_1}(s)\cdots\zeta_{K_r}(s)$$

is $1 + O(p^{-2s})$ for all but finitely many p. Therefore the Dirichlet series for $\widetilde{G}(s)$ is absolutely convergent and nonvanishing on some half-plane $\mathrm{Re}(s) > 1 - \varepsilon$. Writing

$$G(s) = \widetilde{G}(s)\zeta_{K_1}(s)^{-1}\cdots\zeta_{K_r}(s)^{-1}, \tag{20}$$

the continuation of $G(s)$ to $\mathrm{Re}(s) = 1$ and location of zeros along this line now follow from known properties of $\zeta_{K_j}(s)^{-1}$.

The rest of this section is concerned with the demonstration that the derivatives of $G(s)$ can be evaluated by their Dirichlet series along the line $\mathrm{Re}(s) = 1$. Write (20) as $G(s) = \widetilde{G}(s)Z(s)^{-1}$, so $Z(s)^{-1}$ is the product of reciprocal Dedekind zeta functions. When $\mathrm{Re}(s) \geq 1$, the derivative of $G(s)$ to any order is a linear combination of products of a derivative of $\widetilde{G}(s)$ and a derivative of $Z(s)^{-1}$. Since the formally computed Dirichlet series product, at any point, of an *absolutely* convergent Dirichlet series and a convergent Dirichlet series does converge and equal the product, and the Dirichlet series for $\widetilde{G}(s)$ converges absolutely on $\mathrm{Re}(s) > 1 - \varepsilon$, it suffices to prove that the Dirichlet series for all higher derivatives of $Z(s)^{-1}$ converge on $\mathrm{Re}(s) = 1$.

We already know that $Z(s)^{-1}$ is holomorphic on the line $\mathrm{Re}(s) = 1$. The following theorem of Riesz [12, pp. 183–187] gives a method of proving convergence of a Dirichlet series on this line, via analytic continuation: if $h(s) := \sum a_n n^{-s}$ converges for $\mathrm{Re}(s) > 1$ and the average $(1/x)\sum_{n\leq x} a_n$ tends to 0, then $h(s)$ equals $\sum a_n n^{-s}$ at every point on the line $\mathrm{Re}(s) = 1$ to which h admits an analytic continuation. (Convergence of a Dirichlet series anywhere on $\mathrm{Re}(s) = 1$ implies the coefficient average tends to 0, so the hypothesis of Riesz' theorem is natural.)

Repeated differentiation of a series $\sum a_n n^{-s}$ introduces powers of $-\log n$ into the n-th term, so an application of Riesz' theorem to $Z(s)^{-1}$ and its higher derivatives needs an estimate on $(1/x)\sum_{n\leq x} a_n$ which tends to 0 and is stable under replacement of a_n with $a_n \log n$. Such an estimate is

$$\sum_{n\leq x} a_n = O_k\left(\frac{x}{(\log x)^k}\right) \tag{21}$$

for *all* $k > 0$ (the O-constant is allowed to depend on k). When (21) holds for all k, the series $\sum_{n \leq x} a_n \log n$ also satisfies such estimates, by partial summation.

We will prove (21) when $\sum a_n n^{-s}$ is the Dirichlet series for $Z(s)^{-1}$, the product of reciprocal Dedekind zeta functions. That will conclude the proof of Theorem 32.

The following theorem of Selberg and Delange has the conclusion we seek, after we check $Z(s)^{-1}$ and $Z(s)$ satisfy the respective conditions of the Dirichlet series $\sum a_n n^{-s}$ and $\sum b_n n^{-s}$ in the theorem. Write $s = \sigma + it$, as usual.

Theorem 36 (Selberg–Delange) *Let $g(s) = \sum a_n n^{-s}$ converge absolutely for $\mathrm{Re}(s) > 1$. Assume for some positive integer r that $g(s)\zeta(s)^r$ admits an analytic continuation to a region of the form*

$$\sigma \geq 1 - \frac{c}{1 + \log|t|}, \quad |t| \gg 0,$$

where c is a positive constant, and on this region $g(s)\zeta(s)^r$ has absolute value at most a constant times $(1 + |t|)^{1-\delta}$ for some $\delta \in (0, 1)$.

If $|a_n| \leq b_n$, where $(\sum b_n n^{-s})\zeta(s)^{-r}$ has the above properties on the same region, then

$$\sum_{n \leq x} a_n = \frac{x}{(\log x)^{r+1}} O_N\left(\frac{1}{(\log x)^{N+1}}\right)$$

for any $N \geq 0$. In particular, $\sum_{n \leq x} a_n = O_k(x/(\log x)^k)$ for all $k > 0$.

Proof: Using the notation of [13], this is a special case of [13, pp. 185–186] using $z = -r$, which implies that $\sum_{n \leq x} a_n$ is

$$\frac{x}{(\log x)^{r+1}} \left(\sum_{j=0}^{N} \frac{\lambda_j(-r)}{(\log x)^j} + O\left(e^{-c_1\sqrt{\log x}} + \left(\frac{c_2 N + 1}{\log x}\right)^{N+1} \right) \right), \quad (22)$$

where c_1 and c_2 are constants. The definition of $\lambda_j(z)$ in [13] is a sum multiplied by $1/\Gamma(z - j)$. Therefore $\lambda_j(-r) = 0$ for any $j \geq 0$, making the sum in (22) equal to 0. Since $e^{-c_1\sqrt{\log x}} = O_N(1/(\log x)^{N+1})$ for any N, we are done. ∎

It is known that any Dedekind zeta function $\zeta_K(s)$ admits a zero-free region of the shape

$$\sigma \geq 1 - \frac{c_K}{1 + \log|t|}, \quad |t| \gg_K 0 \quad (23)$$

for some constant $c_K > 0$, and on this region $\zeta_K(s)$ and $\zeta_K(s)^{-1}$ are bounded above in absolute value by a power of $\log|t|$. The product $Z(s)^{-1}\zeta(s)^r$, which we already know is holomorphic and nonvanishing on $\mathrm{Re}(s) \geq 1$, involves a finite number of Dedekind zeta functions. Therefore on a region of the shape (23), $Z(s)^{-1}\zeta(s)^r$ and its reciprocal are both nonvanishing and bounded above in absolute value by a power of $\log|t|$. Finally, the Dirichlet coefficients for $Z(s)^{-1}$ are termwise bounded in absolute value by the Dirichlet coefficients of $Z(s)$. The application of Theorem 36 to our situation is now justified, and the proof of Theorem 32 is concluded.

Our proof of Theorem 32 shows $G(s)$ has a zero-free region in the critical strip of the same shape as known zero-free regions for Dedekind zeta functions. If the $\zeta_{K_j}(s)$ in (20) were known to be nonvanishing on a common half-plane $\mathrm{Re}(s) \geq 1 - \delta$ ($\delta > 0$), then the proof of Theorem 32 could be done more directly, without Theorem 36.

8. Concluding Remarks

Although $F_m(s)$ depends *a priori* on m, Theorem 27 shows the right side of (13) is independent of m. This independence of m is a good sign, since we already know by Theorem 19 that $\lim_{s\to 1^+}(s-1)F_m(s)$, if it exists, must not depend on m, so any heuristic which lets us tentatively compute this limit has to give an answer which is independent of m. The interested reader is invited to prove the right side of (13) is independent of m directly, without computing a formula for it which is independent of m as in Theorem 27.

It is reasonable to ask if the existence of $\lim_{s\to 1^+}(s-1)F_m(s)$, which is itself a difficult problem, implies the limit can be correctly computed by a termwise calculation as in (13). In this spirit, we recall some remarks of Lang and Trotter from their work on distributions of Frobenius elements associated to elliptic curves, which is connected to prime values of quadratic polynomials [11, p. 81]. Lang and Trotter write [11, p. 6]

> Again in the case of elliptic curves, can one give a condition on the analytic behavior of the associated Dirichlet series (zeta function) which implies our conjectured asymptotic property? [...] The Hardy–Littlewood paper [7] is in two parts. The first shows how various Riemann hypotheses imply distribution results. The second, including the conjecture on primes in quadratic progressions, limits itself to heuristic arguments. Therefore, even in that case, it would be interesting to see what analytic properties of zeta functions imply the conjectured asymptotic behavior.

Since the Hardy–Littlewood conjecture can be extended to multivariable polynomials over global fields [3], [4], it is worth asking if the arguments of this paper could be extended to these cases. However, it does not seem that the standard Dirichlet series machinery is adequate for such problems about algebraic integers (rather than ideals) in rings other than $\mathbf{Z}$, except perhaps in the ring of integers in an imaginary quadratic field.

Lastly, note that although for $f(T) = T$ the associated Dirichlet series $F(s) = \sum \Lambda(f(n))n^{-s} = -\zeta'(s)/\zeta(s)$ is used in the proof of the prime number theorem, for $f(T) = mT + a$ the associated Dirichlet series $\sum_{n\geq 1} \Lambda(mn+a)n^{-s}$ is not the Dirichlet series $\sum_{n\equiv a \bmod m} \Lambda(n)n^{-s}$ used in the proof of Dirichlet's theorem. Is there a proof of Dirichlet's theorem that uses $\sum_{n\geq 1} \Lambda(mn + a)n^{-s}$ in an essential way?

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References

[1] S. Baier, "On the Bateman–Horn conjecture," *J. Number Theory*, vol. 96, pp. 432–448, 2002.

[2] P. T. Bateman and R. A. Horn, "A heuristic asymptotic formula concerning the distribution of prime numbers," *Math. Comp.*, vol. 16, pp. 363–367, 1962.

[3] B. Conrad, K. Conrad, and R. Gross, "Prime values of one-variable polynomials over global fields" (in preparation).

[4] B. Conrad, K. Conrad, and R. Gross, "Prime values of multivariable polynomials over global fields" (in preparation).

[5] S. Golomb, "Problems in the distribution of prime numbers," Ph.D. thesis (Harvard), 1956.

[6] S. Golomb, "The Lambda method in prime number theory," *J. Number Theory*, vol. 2, pp. 193–198, 1970.

[7] G. H. Hardy and J. E. Littlewood, "Some problems of Partitio Numerorum III: on the expression of a number as a sum of primes," *Acta Math.*, vol. 44, pp. 1–70, 1923.

[8] M. Hindry and J. H. Silverman, *Diophantine Geometry, An Introduction*, Springer-Verlag, New York, 2000.

[9] J. Korevaar, "On Newman's quick way to the prime number theorem," *Math. Intelligencer*, vol. 4, pp. 108–115, 1982.

[10] N. Kurokawa, "Special values of Euler products and Hardy–Littlewood constants," *Proc. Japan Acad. Ser. A Math. Sci.*, vol. 62, pp. 25–28, 1986.

[11] S. Lang and H. Trotter, *Frobenius distributions in* GL_2*-extensions*, Lecture Notes in Mathematics, vol. 504, Spring–Verlag, Berlin, 1976.

[12] M. Riesz, *Collected Papers*, Springer–Verlag, New York, 1988.

[13] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Univ. Press, Cambridge, 1995.