ESTIMATING DEFINITE INTEGRALS OF e^{-t^2} IN TWO WAYS

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The function e^{-t^2} is important since its graph defines the bell curve in probability theory[1], which relates the area under its graph to the normal distribution. However, e^{-t^2} has no concrete antiderivative, so $\int_a^b e^{-t^2}\,dt$ when $a < b$ is found by approximations.

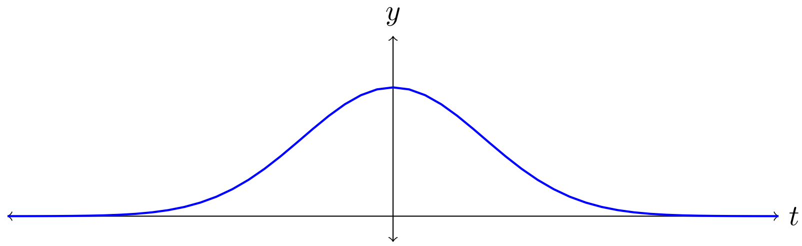

FIGURE 1. Plot of $y = e^{-t^2}$.

We will approximate $\int_0^T e^{-t^2}\,dt$ when $T > 0$ in two ways, power series and integration by parts, and apply each to estimate $\int_0^3 e^{-t^2}\,dt$. The methods are taken from a 19th century astronomy book [1, pp. 154-155], where $\int_0^T e^{-t^2}\,dt$ is used in calculations related not to probability, but to light refraction (how light rays pass through our atmosphere). We will bound errors in these approximations as in [1], where the author writes in the preface

> When approximate methods are employed . . . their degree of accuracy is carefully determined by . . . converging series, and only such terms [are] neglected as can be shown to be insensitive in the cases to which the formulas are to be applied.

Method 1: Series.

Since $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$ for all x, $e^{-t^2} = \sum_{n\geq 0} \frac{(-t^2)^n}{n!} = \sum_{n\geq 0} \frac{(-1)^n}{n!} t^{2n}$. By termwise integration,

$$\begin{aligned}
\int_0^T e^{-t^2}\,dt &= \int_0^T \left(\sum_{n\geq 0} \frac{(-1)^n}{n!} t^{2n}\right) dt \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_0^T t^{2n}\,dt \\
&= \sum_{n\geq 0} \frac{(-1)^n T^{2n+1}}{n!(2n+1)} \qquad (1) \\
&= T - \frac{T^3}{3} + \frac{T^5}{2\cdot 5} - \frac{T^7}{6\cdot 7} + \frac{T^9}{24\cdot 9} - \frac{T^{11}}{120\cdot 11} + \cdots .
\end{aligned}$$

[1]The standard bell curve is actually the graph of $\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ rather than e^{-t^2}, but we'll work with e^{-t^2} to avoid extra constant factors in our calculations.

When $T > 0$ this is an alternating series whose successive terms decrease in absolute value, so

$$\boxed{\int_0^T e^{-t^2}\,dt = \sum_{n=0}^{N} \frac{(-1)^n T^{2n+1}}{n!(2n+1)} + R_N,}$$

where

$$|R_N| \le \left|\frac{(-1)^{N+1} T^{2(N+1)+1}}{(N+1)!(2(N+1)+1)}\right| = \frac{T^{2N+3}}{(N+1)!(2N+3)}.$$

Take $T = 3$, so $\int_0^3 e^{-t^2}\,dt = s_N + R_N$, where $s_N = \sum_{n=0}^{N} \frac{(-1)^n 3^{2n+1}}{n!(2n+1)}$ and $|R_N| \le \frac{3^{2N+3}}{(N+1)!(2N+3)}$. The table below presents these values when $N = 10$, 20, 30, and 40.

N	s_N	$\lvert R_N\rvert$ bound
10	60.7	1127.9
20	.993	3.1377
30	.88620908	6.8491
40	.886207348260	$5.89 \cdot 10^{-11}$

Using the error bound when $N = 40$,

$$.88620734820 < \int_0^3 e^{-t^2}\,dt < .88620734832, \tag{2}$$

which gives us the first 9 decimal places in the integral: .886207348.

Method 2: Integration by parts.

Apply integration by parts to $\int_0^T e^{-t^2}\,dt$ with $u = e^{-t^2}$ and $dv = dt$, so $du = -2te^{-t^2}\,dt$ and $v = t$:

$$\int_0^T e^{-t^2}\,dt = uv\,\Big|_{t=0}^{t=T} - \int_0^T v\,du = Te^{-T^2} + 2\int_0^T t^2 e^{-t^2}\,dt. \tag{3}$$

Use integration by parts again in the new integral, with $u = e^{-t^2}$ as before and $dv = t^2\,dt$, so $du = -2te^{-t^2}\,dt$ and $v = t^3/3$:

$$\int_0^T t^2 e^{-t^2}\,dt = uv\,\Big|_{t=0}^{t=T} - \int_0^T v\,du = \frac{T^3}{3}e^{-T^2} + \frac{2}{3}\int_0^T t^4 e^{-t^2}\,dt.$$

Substituting this into (3),

$$\begin{aligned}\int_0^T e^{-t^2}\,dt &= Te^{-T^2} + 2\left(\frac{T^3}{3}e^{-T^2} + \frac{2}{3}\int_0^T t^3 e^{-t^2}\,dt\right)\\ &= Te^{-T^2} + \frac{2}{3}T^3 e^{-T^2} + \frac{2^2}{3}\int_0^T t^4 e^{-t^2}\,dt. \qquad (4)\end{aligned}$$

We've faced $\int_0^T e^{-t^2}\,dt$, $\int_0^T t^2 e^{-t^2}\,dt$, and now $\int_0^T t^4 e^{-t^2}\,dt$. Let's just consider the general integral

$$I_n = \int_0^T t^{2n} e^{-t^2}\,dt$$

with an even exponent $2n$, where $n \geq 0$. Use integration by parts on this with $u = e^{-t^2}$ and $dv = t^{2n}\,dt$, so $du = -2te^{-t^2}\,dt$ and $v = t^{2n+1}/(2n+1)$:

$$I_n = \int_0^T t^{2n}e^{-t^2}\,dt = uv\,\Big|_{t=0}^{t=T} - \int_0^T v\,du = \frac{T^{2n+1}}{2n+1}e^{-T^2} + \frac{2}{2n+1}\int_0^T t^{2(n+1)}e^{-t^2}\,dt,$$

so

$$I_n = \frac{T^{2n+1}}{2n+1}e^{-T^2} + \frac{2}{2n+1}I_{n+1}. \tag{5}$$

We'll use this recursion to recover (3) and (4) and go further:

$$\begin{aligned}
\int_0^T e^{-t^2}\,dt &= I_0 \\
&= Te^{-T^2} + 2I_1 \\
&= Te^{-T^2} + 2\left(\frac{T^3}{3}e^{-T^2} + \frac{2}{3}I_2\right) \\
&= Te^{-T^2} + \frac{2}{3}T^3e^{-T^2} + \frac{2^2}{3}I_2 \\
&= Te^{-T^2} + \frac{2}{3}T^3e^{-T^2} + \frac{2^2}{3}\left(\frac{T^5}{5}e^{-T^2} + \frac{2}{5}I_3\right) \\
&= Te^{-T^2} + \frac{2}{3}T^3e^{-T^2} + \frac{2^2}{3\cdot 5}T^5e^{-T^2} + \frac{2^3}{3\cdot 5}I_3 \\
&= Te^{-T^2} + \frac{2}{3}T^3e^{-T^2} + \frac{2^2}{3\cdot 5}T^5e^{-T^2} + \frac{2^3}{3\cdot 5}\left(\frac{T^7}{7}e^{-T^2} + \frac{2}{7}I_4\right) \\
&= Te^{-T^2} + \frac{2}{3}T^3e^{-T^2} + \frac{2^2}{3\cdot 5}T^5e^{-T^2} + \frac{2^3}{3\cdot 5\cdot 7}T^7e^{-T^2} + \frac{2^4}{3\cdot 5\cdot 7}I_4 \\
&= Te^{-T^2} + \frac{2}{3}T^3e^{-T^2} + \frac{2^2}{3\cdot 5}T^5e^{-T^2} + \frac{2^3}{3\cdot 5\cdot 7}T^7e^{-T^2} + \frac{2^4}{3\cdot 5\cdot 7}\left(\frac{T^9}{9}e^{-T^2} + \frac{2}{9}I_5\right) \\
&= Te^{-T^2} + \frac{2}{3}T^3e^{-T^2} + \frac{2^2}{3\cdot 5}T^5e^{-T^2} + \frac{2^3}{3\cdot 5\cdot 7}T^7e^{-T^2} + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}T^9e^{-T^2} + \frac{2^5}{3\cdot 5\cdot 7\cdot 9}I_5.
\end{aligned}$$

The general pattern here is

$$\boxed{\int_0^T e^{-t^2}\,dt = e^{-T^2}\sum_{n=0}^{N}\frac{2^n}{1\cdot 3\cdot 5\cdots(2n+1)}T^{2n+1} + \widetilde{R}_N,} \tag{6}$$

where the sum over $0 \leq n \leq N$ has all positive terms (it is not an alternating series) and

$$\widetilde{R}_N := \frac{2^{N+1}}{1\cdot 3\cdot 5\cdots(2N+1)}I_{N+1} = \frac{2^{N+1}}{1\cdot 3\cdot 5\cdots(2N+1)}\int_0^T t^{2(N+1)}e^{-t^2}\,dt.$$

The remainder $\widetilde{R}_N$ is positive. To bound it from above, let's bound the integrand $t^{2(N+1)}e^{-t^2}$ on $[0,T]$. Call that integrand $f_N(t)$. It vanishes at $t = 0$, is positive when $t > 0$, and a calculation left to the reader shows

$$f_N'(t) = 2t^{2N+1}e^{-t^2}(N+1-t^2),$$

so $f_N'(t) > 0$ when $0 < t < \sqrt{N+1}$, $f_N'(\sqrt{N+1}) = 0$, and $f_N'(t) < 0$ when $t > \sqrt{N+1}$. Thus f_N is increasing when $0 < t < \sqrt{N+1}$ and is maximized at $t = \sqrt{N+1}$, so when $T \leq \sqrt{N+1}$ (equivalently, $N+1 \geq T^2$) and $0 \leq t \leq T$, we have

$$0 \leq f_N(t) \leq f_N(T) = T^{2(N+1)}e^{-T^2}.$$

Therefore

$$N+1 \geq T^2 \Longrightarrow \int_0^T t^{2(N+1)}e^{-t^2}\,dt \leq T^{2(N+1)}e^{-T^2}\int_0^T dt = T^{2N+3}e^{-T^2},$$

so

$$\begin{aligned}\widetilde{R}_N &= \frac{2^{N+1}}{1\cdot 3\cdot 5\cdots(2N+1)}\int_0^T t^{2(N+1)}e^{-t^2}\,dt \\ &\leq \frac{2^{N+1}}{3\cdot 5\cdots(2N+1)}T^{2N+3}e^{-T^2} \\ &= Te^{-T^2}\frac{(2T^2)^{N+1}}{1\cdot 3\cdot 5\cdots(2N+1)}\end{aligned}$$

when $N+1 \geq T^2$. This bound on $\widetilde{R}_N$ (with fixed T) tends to 0 as $N \to \infty$, so letting $N \to \infty$ in (6) gives us a second formula for $\int_0^T e^{-t^2}\,dt$ using an infinite series of positive terms with a factor e^{-T^2} out front:

$$\begin{aligned}\int_0^T e^{-t^2}\,dt &= e^{-T^2}\sum_{n\geq 0}\frac{2^n}{1\cdot 3\cdot 5\cdots(2n+1)}T^{2n+1} \\ &= e^{-T^2}\left(T + \frac{2}{3}T^3 + \frac{2^2}{3\cdot 5}T^5 + \frac{2^3}{3\cdot 5\cdot 7}T^7 + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}T^9 + \frac{2^5}{3\cdot 5\cdot 7\cdot 9\cdot 11}T^{11} + \cdots\right).\end{aligned}$$

Compare this to (1), which is an alternating series.

In (6) set $T = 3$, so $\int_0^3 e^{-t^2}\,dt = \widetilde{s}_N + \widetilde{R}_N$, where $\widetilde{s}_N = e^{-9}\sum_{n=0}^{N}\frac{2^n}{1\cdot 3\cdot 5\cdots(2n+1)}3^{2n+1}$ and $0 < \widetilde{R}_N \leq 3e^{-9}\frac{18^{N+1}}{1\cdot 3\cdot 5\cdots(2N+1)}$. The table below has these values when $N = 10$, 20, 30, and 40 (so $N+1 \geq 3^2 = 9$).

N	$\widetilde{s}_N$	$\widetilde{R}_N$ bound
10	.6712	1.7305
20	.885960	.0064787
30	.8862073445	$1.702\cdot 10^{-7}$
40	.886207348259518	$1.676\cdot 10^{-13}$

Using the error bound when $N = 40$,

$$.886207348259350 < \int_0^3 e^{-t^2}\,dt < .886207348259685,$$

which gives us the first 12 decimal places in the integral: .886207348259. Our previous estimate, with $N = 40$ in (2), gave us the first 9 decimal places in this integral.

References

[1] W. Chauvenet, *A Manual of Spherical and Practical Astronomy*, Volume 1, J. B. Lippincott & Co., Philadelphia, 1863. URL https://books.google.com/books?id=JeokvgAACAAJ.